33A Linear Algebra: Homework 9

Puck Rombach

Warm-up exercises

Due: 6/3/16

Problem 1

Students in a mathematics lecture are asked each lecture whether they are happy or unhappy. A transition diagram is drawn below.

Let the vector \( \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \) show the probability distribution of students happiness on a given day \( t \). This means that a proportion \( x_1(t) \) of students are happy and \( x_2(t) \) are unhappy, such that \( x_1(t) + x_2(t) = 1 \), and \( x_1(t), x_2(t) \geq 0 \).

(a) Give the transition matrix \( A \) that corresponds to this process \( \vec{x}(t) = A \vec{x}(t - 1) \).

(b) Find the probability that a student is unhappy in lecture 3, if she was happy in lecture 1.

(c) Show that, on average, 1 out of 4 students is unhappy.

Problem 2

Find all matrices with eigenvectors \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \), with eigenvalues \( \sqrt{2} \) and \( \cos(\pi/3) \), respectively.

Problem 3

This might look a little tricky, but you can do it. Suppose our dynamical system requires us to add constants to our recurrence relations. For example, we would like to model a system governed by

\[
\begin{align*}
x_1(t) &= x_1(t - 1) + x_2(t - 1) + 1 \\
x_2(t) &= x_1(t - 1) - x_2(t - 1) + 3.
\end{align*}
\]

Start by finding a \( 2 \times 2 \) matrix \( A \) and a \( \vec{b} \in \mathbb{R}^2 \) so you can write this as a relation \( \vec{x}(t) = A \vec{x}(t - 1) + \vec{b} \).

Then, instead, start using a vector in \( \mathbb{R}^3 \), but with \( x_3 = 1 \) always. Now, \( \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ 1 \end{pmatrix} \). Now, is it possible to find a \( 3 \times 3 \) matrix \( A \) such that \( \vec{x}(t) = A \vec{x}(t - 1) \).
Hand-in Exercise

Problem 1  The Fibonacci Sequence

The golden ratio and the Fibonacci sequence are beautiful mathematical concepts that appear frequently in nature, geeky tattoos and bad science. We will start by defining the Fibonacci sequence, and then use linear algebra to find the golden ratio. The Fibonacci numbers are defined recursively as follows:

\[ f(n) = f(n-1) + f(n-2), \quad f(0) = 0, f(1) = 1. \]  

(1)

Every number in the sequence is the sum of the previous two numbers in the sequence, and we start with 0 and 1. So, the first few numbers in the Fibonacci sequence are

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

If we record these numbers as vectors consisting of two consecutive Fibonacci numbers, then we can find the next vector as a function of the current one. This means that we can write the sequence as a discrete dynamical system. More precisely, we can use the system of linear equations that consists of Equation (1) and the trivial equation

\[ f(n-1) = f(n-1). \]

We then use

\[ \vec{f}_n = \begin{pmatrix} f(n) \\ f(n-1) \end{pmatrix}, \quad \vec{f}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]

Now, the first few vectors in our sequence are

\[ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 8 \\ 5 \end{pmatrix}, \begin{pmatrix} 13 \\ 8 \end{pmatrix}, \begin{pmatrix} 21 \\ 13 \end{pmatrix}, \begin{pmatrix} 34 \\ 21 \end{pmatrix}, \begin{pmatrix} 55 \\ 34 \end{pmatrix}, \cdots \]

Look back and forth a few times between the scalar sequence and the vector sequence until you feel not confused by it. Then answer the following questions.

(a) Find the matrix \( A \) of the Fibonacci sequence, such that \( \vec{f}(n) = A \vec{f}(n-1), \) with \( n > 1 \) and \( \vec{f}(1) \) as given above.

(b) Find the eigenvectors \( \vec{v}_1, \vec{v}_2 \) and corresponding eigenvalues \( \lambda_1, \lambda_2 \) (ordered such that \( \lambda_1 > \lambda_2 \)) of \( A \), and show that the eigenvectors can be written as \( \vec{v}_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} \) and \( \vec{v}_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix} \). You may use the following fact: \[ \frac{2}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{2}. \]

(c) Find \( B \) and \( S, S^{-1} \) such that \( B \) is diagonal and \( A = SBS^{-1} \).

(d) Use \( S^{-1} \) from (c) to find \( a_1 \) and \( a_2 \) such that \( \vec{f}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = a_1 \vec{v}_1 + a_2 \vec{v}_2 \).

(e) Write a direct formula for the Fibonacci numbers of the form

\[ \vec{f}_n = a_1 \lambda_1^{n-2} \vec{v}_1 + a_2 \lambda_2^{n-2} \vec{v}_2. \]

(The reason for the \( n - 2 \) in the exponent is that we are starting at \( \vec{f}_2 \) as found in (d) as our starting point.).
(f) Sketch the trajectory of the system in \( \mathbb{R}^2 \) in a way that shows the limit behavior, and compare this to the phase portrait of a discrete system which has eigenvectors \( \vec{v}_1, \vec{v}_2 \) and eigenvalues \( |\lambda_1|, |\lambda_2| \).

(g) Explain why, as \( n \to \infty \), we have
\[
\lim_{n \to \infty} \frac{f_n \cdot \vec{v}_2}{f_n \cdot \vec{v}_1} = 0.
\]

(h) Explain what happens to the ratio \( f(n)/f(n-1) \) as \( n \) grows to infinity.

Optional Exercise

Problem 1

A Padovan sequence is defined in a similar way to the Fibonacci sequence. Every entry is defined as a sum of two previous entries, but instead of adding up the previous entry and the one before that, one takes the entry before the previous and the one before that. So, the sequence is defined as:
\[
P(1) = P(2) = P(3) = 1, \quad P(n) = P(n-2) + P(n-3).
\]
The first few values in the sequence are
\[
1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, \ldots
\]

(a) Give the next two values in the Padovan sequence.

(b) We can define this sequence as a discrete dynamical system, where we write
\[
\vec{P}(3) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\]
\[
\vec{P}(n) = \begin{pmatrix} P(n) \\ P(n-1) \\ P(n-2) \end{pmatrix}
\]

and
\[
\vec{P}(n) = A \vec{P}(n-1),
\]
for \( n > 3 \). Give the matrix \( A \) that defines this system.

(c) Show that the Padovan sequence gives us an analogue to the golden ratio emerging from the Fibonacci sequence.

(d) According to Padovan/vd Laan, the number you found in (c), called the “plastic number”, is intricately linked to the ratios 3/4 and 1/7 (which they again like for aesthetic reasons). Where did those ratios come from? I’m mystified.