Problem 1

Play around with the demonstration project at goo.gl/bLkej0. Then use it to find $\vec{x}_B$, without doing any calculations, where $\vec{x} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $B = \begin{pmatrix} 4 \\ -5 \end{pmatrix}, \begin{pmatrix} 4 \\ -4 \end{pmatrix}$. Sketch the picture. (If you have trouble hitting the integers just accept a small error in your picture.)

Also try the demonstration project at goo.gl/hM47NN.

Problem 2

Show that every $2 \times 2$ matrix that represents a projection onto a line $L$ is similar to $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Problem 3

Is matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ similar to $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ for all $a$ and $b$?

Problem 4

What is the relationship between the eigenvectors and eigenvalues of a matrix and its transpose?

Problem 5

If two matrices have the same characteristic polynomial, which of the following do they necessarily have in common: rref, invertibility, eigenvalues, eigenbasis, geometric interpretation? Are they similar?
Hand-in Exercises

Problem 1

If 3 is an eigenvalue of $A$ and of $B$, must 9 be an eigenvalue of $AB$? (Show why it is true or give a counter-example.)

Problem 2

Suppose a real $3 \times 3$ matrix $A$ has only two distinct eigenvalues. If $\text{tr}(A) = 7$ and $\det(A) = 5$, find at least 1 possible set of eigenvalues and their algebraic multiplicities. (How many are there? You may use a computer to find them.) What do we know about their geometric multiplicities?

Problem 3

Find the eigenspaces of the $2 \times 2$ real matrix $A$ representing a $\pi/4$ rotation in the plane, and diagonalize $A$ over $\mathbb{C}$.

Problem 4

Consider the plane $V$ in $\mathbb{R}^3$ spanned by the equation $x_1 - x_2 - x_3 = 0$. Let $\vec{x} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. Find a basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$ of $V$, such that $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Problem 5

If $A$ is the $n \times n$ matrix representing the projection onto the subspace $V \subset \mathbb{R}^n$, with $\dim(V) = m$, and $B$ represents reflection in $V$, then what do you know about the traces and determinants of $A$ and $B$?
Optional Exercises

Problem 1

Suppose that a $2 \times 2$ matrix $A$ has eigenvectors $\vec{v}_1$ and $\vec{v}_2$, with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{3}$. What is $\lim_{k \to \infty} A^k \vec{x}$, for $\vec{x} \in \mathbb{R}^2$?

Problem 2

Let $A$ be a square matrix such that the sum of each column is 1. Show that 1 is an eigenvalue of $A$.

Problem 3

Have a look at the following discussion on Game Development stackexchange: goo.gl/h3ZBgj. Consider the following tesselation of the plane (suggested in that thread), that uses squares and octagons. Note that all line segments have the same length. We call the points where line segments meet the vertices.

I have indicated a basis $\mathcal{B} = (\vec{v}, \vec{w})$ for the plane. Now, we can express any vector as a linear combination of $\vec{v}$ and $\vec{w}$. We consider vectors between pairs of vertices $O, P, Q, R$. (We only care about the magnitude/direction of vectors, not about where they originate.) For example, the vector $\vec{OR}$, that gets you from point $O$ to point $R$, can be expressed as $\vec{OR} = -\vec{v}$. So, $[\vec{OR}]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Can you find $[\vec{OP}]_{\mathcal{B}}$ and $[\vec{OQ}]_{\mathcal{B}}$?

If instead of using a strict basis of $\mathbb{R}^2$ to move between vertices, you were allowed to use a spanning set that is not a basis to get around (paying the price that the coordinate vectors are not unique anymore), which set would you use?