Question 1

Use a counting argument to prove the following identity:

\[ \sum_{i=m}^{n} \binom{n}{i} \binom{i}{m} = \binom{n}{m} 2^{n-m}. \]

(Count ways to pick two disjoint subsets from a set of size \( n \).)

Answer

We pick two disjoint subsets \( S_1, S_2 \) from a set of cardinality \( n \), with \( |S_1| = m \) and \( 0 \leq |S_2| \leq n - m \). We count all possible such pairs \((S_1, S_2)\).

One way to pick our subsets \( S_1, S_2 \) is to first pick a subset \( S_1 \cup S_2 \) of cardinality \( m \leq |S_1 \cup S_2| \leq n \), and then pick a subset of cardinality \( m \) from this larger set which will be \( S_1 \). Clearly, each choice of large set and subsequent splitting gives a unique pair \((S_1, S_2)\). This gives us a total of \( \sum_{i=m}^{n} \binom{n}{i} \binom{i}{m} \) pairs.

A second method to pick our pair \((S_1, S_2)\) is to first select \( m \) elements to form \( S_1 \), and then select any subset of the remaining elements to form \( S_2 \). This can be done in \( \binom{n}{m} 2^{n-m} \) ways. Together, we obtain

\[ \sum_{i=m}^{n} \binom{n}{i} \binom{i}{m} = \binom{n}{m} 2^{n-m}. \]

Question 2

I pick 5 cards uniformly at random from a deck of 52 cards (13 different values for 4 different suits). I tell you that the first card I drew was a 4 of hearts. What is the probability that my hand contains 4 of a kind? (Four cards of the same value in different suits.)

Answer

You may notice that there is a high degree of symmetry between the cards. For example, we can relabel the suits or the values without changing the problem (since the question does not ask for four of a kind of any particular value). Therefore, this information does not change the probability of getting a four of a kind.

Let’s work this out explicitly. Let \( A \) be the event that our hand contains four of a kind. Let \( B \) be the event that the first card is a 4 of hearts. Then, by the definition of conditional probability, we have

\[ \mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}. \]

We have \( \mathbb{P}(B) = 1/52 \). For, \( \mathbb{P}(A \cap B) \), we count the number of ways in which we can pick the remaining four cards in a way that creates a four of a kind hand. If our 4 of hearts is not part of the four of a kind,
then there are 12 ways to pick the remaining four cards (ignoring their order). If the 4 of hearts is part of the four of a kind, then there are 48 ways to pick the remaining four cards (the only choice is the one that determines the card that is not a 4). In total, this gives,

\[ \mathbb{P}(A \cap B) = \frac{1}{52} \cdot \frac{12 + 48}{\binom{51}{4}}. \]

Therefore, we have

\[ \mathbb{P}(A|B) = \frac{60}{\binom{51}{4}}. \]

(This is indeed equal to \( \frac{13 \cdot 48}{\binom{52}{5}} \), the unconditional probability of four of a kind, although that is a little hard to work out by hand.)

**Question 3**

Find a recurrence relation for \( c(n) \) which counts the number of bitstrings of length \( n \) (\( \{0, 1\}^n \)) that contain at least one pair of consecutive 1s ("11"). You don’t need to give the initial conditions.

(Hint: The relation will look like \( c(n) = c(\ldots) + c(\ldots) + 2^{n-1} \).)

**Answer**

Suppose that we know the values of \( c(k) \) for \( 2 \leq k < n \), and we want to find the value for \( c(n) \). We partition the sequences into three groups: sequences that start with 0, sequences that start with 10, and sequences that start with 11. This is a proper partition since each sequence of length \( \leq 2 \) falls into exactly one of these classes. We count each class separately.

- If a sequence starts with 0, then any occurrence of 11 will fall entirely in the last \( n-1 \) bits. Therefore, there are \( c(n-1) \) such sequences.

- If a sequence starts with 10, then any occurrence of 11 will fall entirely in the last \( n-2 \) bits. Therefore, there are \( c(n-2) \) such sequences.

- If a sequence starts with 11, it is guaranteed to contain 11. Therefore, the remaining \( n-2 \) bits may be anything. There are \( 2^{n-2} \) such sequences.

This gives a total of

\[ c(n) = c(n-1) + c(n-2) + 2^{n-2}. \]