Question 1

(a) [2 points] Define the eigenspace $E_\lambda$ of a square matrix $A$, if $\lambda$ is an eigenvalue.

(b) [2 points] If $\vec{v}$ is an eigenvector of $A$ and of $B$ (with eigenvalues $\lambda_A$ and $\lambda_B$ respectively), is $\vec{v}$ also an eigenvector of $A + B$? If so, what is the eigenvalue?

(c) [2 points] If $\vec{v}$ is an eigenvector of $A$ (with eigenvalue $\lambda_A$), is $A\vec{v}$ also an eigenvector of $A$? If so, what is the eigenvalue?

(d) [4 points] Find the eigenvalues of the following matrix, together with their algebraic and their geometric multiplicities:

$$A = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix}.$$ 

Answer.

(a) Possible definitions:

- The eigenspace $E_\lambda$ is the space spanned by the eigenvectors of $A$ with eigenvalue $\lambda$.
- The eigenspace $E_\lambda$ is the set of all eigenvectors of $A$ with eigenvalue $\lambda$, and the vector $\vec{0}$.
- $E_\lambda = \ker(A - \lambda I_n)$.

(b) We have $(A + B)\vec{v} = A\vec{v} + B\vec{v} = \lambda_A \vec{v} + \lambda_B \vec{v} = (\lambda_A + \lambda_B)\vec{v}$. So, $\vec{v}$ is an eigenvector of $A + B$ with eigenvalue $\lambda_A + \lambda_B$.

(c) We have $A(A\vec{v}) = A(\lambda_A \vec{v}) = \lambda_A (A\vec{v})$. So, $A\vec{v}$ is an eigenvector of $A$ with eigenvalue $\lambda_A$.

(d) We have

$$f_A(\lambda) = (-\lambda)(1 - \lambda)(-2 - \lambda) - 3(1 - \lambda) = (1 - \lambda)(\lambda^2 + 2\lambda - 3) = (1 - \lambda)(\lambda - 1)(\lambda + 3).$$

So, we get eigenvalues 1 and $-3$, with $\text{almu}(1) = 2$ and $\text{almu}(-3) = 1$. To find the geometric multiplicities, we have to look at the dimensions of the eigenspaces:

$$E_1 = \ker(A - I_n) = \ker\begin{pmatrix} -1 & 0 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & -3 \end{pmatrix} = \ker\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{span}\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix},$$

$$E_{-3} = \ker(A + 3I_n) = \ker\begin{pmatrix} 3 & 0 & 3 \\ 0 & 4 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \ker\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{span}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

So, we have $\text{gemu}(1) = 1$ and $\text{gemu}(-3) = 1$. 
Question 2

For each of the following statements, explain whether they are true or false (in general, for any $n \times m$ matrices $A, B$).

(a) [2 points] $\text{im}(A) \cup \text{ker}(A^T) = \mathbb{R}^n$

(b) [2 points] $\text{ker}(A) = \text{ker}(A^T A)$

(c) [2 points] $(A + B)^T = A^T + B^T$

(d) [2 points] $\det(A) = \det(A^T)$ (if $n = m = 2$).

Answer.

(a) False. The subspace $\text{im}(A)$ is the orthogonal complement of $\text{ker}(A^T)$. Orthogonal complements do not cover all of $\mathbb{R}^n$. For example, take two nonzero vectors $\vec{v} \in \text{im}(A)$ and $\vec{w} \in \text{ker}(A^T)$. Clearly $\vec{v} \cdot \vec{w} = 0$. Then the vector $\vec{x} = \vec{v} + \vec{w}$ is in neither of the spaces $\text{im}(A)$ or $\text{ker}(A^T)$, because it is orthogonal to neither of them ($\vec{x} \cdot \vec{v} = ||\vec{v}||^2$ and $\vec{x} \cdot \vec{w} = ||\vec{w}||^2$).

(b) True. If $\vec{x} \in \text{ker}(A)$, then $\vec{x} \in \text{ker}(A^T A)$. Any $A\vec{x} \neq \vec{0}$ in the image of $A$ is not in $\text{ker}(A^T)$. So, if $\vec{x} \notin \text{ker}(A)$, then $\vec{x} \notin \text{ker}(A^T A)$. Therefore, $\text{ker}(A) = \text{ker}(A^T A)$.

(c) True. We have $[(A + B)^T]_{ij} = [A + B]_{ji} = [A]_{ji} + [B]_{ji} = [A^T]_{ij} + [B^T]_{ij} = [A^T + B^T]_{ij}$.

(d) True. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Then $\det(A) = ad - cb = ad - bc = \det(A^T)$. 

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Question 3

Let \( \hat{u}_1, \hat{u}_2 \) be an orthonormal (with respect to the dot product) basis of \( \mathbb{R}^2 \).

(a) [2 points] Is it possible for any of the vectors \( \hat{u}_i \) to have an entry \( > 1 \)?

(b) [3 points] Suppose that \( \vec{x} = c_1 \hat{u}_1 + c_2 \hat{u}_2 \), and we also know that \( \vec{x} \) is orthogonal to \( L = \text{span}(\hat{u}_1) \).
Show carefully that \( c_1 = 0 \).

(c) [5 points] For \( \vec{v}, \vec{w} \in \mathbb{R}^2 \), let \( \langle \vec{v}, \vec{w} \rangle = T(\vec{v}) \cdot T(\vec{w}) \), where
\[
T(\vec{v}) = A\vec{v} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \vec{v}.
\]
Show that the standard basis is not orthogonal with respect to this inner product, and find a basis of \( \mathbb{R}^2 \) which is.

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Answer.

(a) We know that
\[
\|\hat{u}_i\|^2 = 1 = [\hat{u}_i]^1_1 + [\hat{u}_i]^2_2.
\]
This implies both entries must be \( \leq 1 \).

(b) We have \( 0 = \vec{x} \cdot \hat{u}_1 = c_1 \hat{u}_1 \cdot \hat{u}_1 + c_2 \hat{u}_2 \cdot \hat{u}_1 = c_1 \) (from the orthonormality).

(c) We have
\[
\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \cdot \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = 1,
\]
which shows that the standard basis is not orthogonal with respect to this inner product. This transformation is a shear followed by a regular dot product, so to find a basis that is orthogonal, we need to find a set of vectors that is orthogonal after applying the shear. For example, the set
\[
\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}
\]
works:
\[
\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle = \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \cdot \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0.
\]
We can also take a different approach to this question, and rewrite this inner product as
\[
\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle = (a + b)(c + d) + cd.
\]
Setting this to \( 0 \) gives us a lot of freedom!