1. Find the (absolute) maximum and minimum values of 

\[ f(x) = (x^2 - 2)^{2/3} \]

on the interval \([-1, 3]\), showing your work.

\[
\frac{d}{dx} f(x) = \frac{2}{3} (x^2 - 2)^{-1/3} (2x) = \frac{4x}{3(x^2 - 2)}
\]

critical numbers: \(-1, 0, \sqrt{2}, 3\)

\[ f(-1) = (-1)^{2/3} = 1 \]
\[ f(0) = (-2)^{2/3} = \sqrt[3]{4} \]
\[ f(\sqrt{2}) = 0 \quad \text{min} \]
\[ f(3) = 7^{2/3} = \sqrt[3]{49} \quad \text{max} \]
\[ (x^2 + y^2)^2 = 2(x^2y + 1) \]

at \((-1, 1).\)

\[
2(x^2 + y^2)(2x + 2y \frac{dy}{dx}) = 2(2xy + x^2 \frac{dy}{dx})
\]

\[
(1 + 1)(-2 + 2 \frac{dy}{dx}) = -2 + \frac{dy}{dx}
\]

\[-4 + 4 \frac{dy}{dx} = -2 \frac{dy}{dx} \quad \frac{dy}{dx} = \frac{2}{3} \]

\[ y - 1 = \frac{2}{3} (x + 1) \]
3. A tank is in the shape of an inverted cone of height 4m and radius 3m at the top. Water is being pumped into the tank at a constant rate of 7m³/min. However, water is also leaking out of the tank at a constant rate. If the depth of the water is rising at a rate of 8m/min when the water level is \( \frac{2}{3} \)m, at what rate is water leaking from the tank? Note: the volume of a cone of radius \( r \) and height \( h \) is \( \frac{1}{3} \pi r^2 h \).

\[
\begin{align*}
\frac{h}{r} &= \frac{4}{3} \\
\frac{r}{h} &= \frac{2}{4} = \frac{1}{2} \\
V &= \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi \left( \frac{2}{3} h \right)^2 h \\
&= \frac{3 \pi}{16} h^3
\end{align*}
\]

\[
\frac{dV}{dt} = \frac{3 \pi}{16} h^2 \frac{dh}{dt}
\]

\[
7 - C = \frac{9 \pi}{16} \left( \frac{2}{3} \right)^2 (8) = 2 \pi
\]

rate of leak = \( C = 7 - 2 \pi \) m³/min
4. The function \( f(x) = \sqrt{x^2 + 3x} + x \)

has a horizontal asymptote. Find it, showing your work.

\[
\lim_{x \to -\infty} \sqrt{x^2 + 3x} + x = \lim_{x \to -\infty} \sqrt{(-x)^2 + 3(-x)} + (-x)
\]

\[
= \lim_{x \to -\infty} \sqrt{x^2 - 3x} - x \cdot \frac{\sqrt{x^2 - 3x} + x}{\sqrt{x^2 - 3x} + x}
\]

\[
= \lim_{x \to -\infty} \frac{(x^2 - 3x) - x^2}{\sqrt{x^2 - 3x} + x}
\]

\[
= \lim_{x \to -\infty} \frac{-3x}{\sqrt{x^2 - 3x} + x} \cdot \frac{1}{x}
\]

\[
= \lim_{x \to -\infty} \frac{-3}{\sqrt{1 - \frac{3}{x}} + 1} = -\frac{3}{2}
\]
5. Let \( f(x) = ax^3 + bx^2 + cx + d \)
be the general cubic polynomial where \( a, b, c \) and \( d \) are constants and \( a \neq 0 \).

(10 points) (a) Prove that \( f(x) \) has a point of inflection.

(10 points) (b) Prove that if \( x = 0 \) and \( x = 1 \) are critical numbers of \( f(x) \),
then the point of inflection is \( x = \frac{1}{2} \).

(a) \[ f'(x) = 3ax^2 + 2bx + c \]
\[ f''(x) = 6ax + 2b = 0 \]
\[ x = -\frac{b}{3a} \text{ is a point of inflection} \]

(b) \[ f'(0) = c = 0 \]
\[ f'(1) = 3a(1) + 2b(1) + 0 = 0 \]
\[ b = -\frac{3}{2}a \]
So the point of inflection is \[ x = -\frac{b}{3a} = -\frac{(-\frac{3}{2}a)}{3a} = \frac{1}{2} \]
(part (a))