1. Given the function

\[ f(x) = x^4 - 2x^3 - 12x^2 + 5 \]

determine where the function is concave upward and concave downward, and find the points of inflection, if there are any.

\[ f'(x) = 4x^3 - 6x^2 - 24x \]

\[ f''(x) = 12x^2 - 12x - 24 \]

\[ = 12(x^2 - x - 2) \]

\[ = 12(x - 2)(x + 1) \]

\[ f''(x) = 0 \text{ if } x = -1, 2 \]

\[
\begin{array}{cccc}
\text{x - 2} & \text{x + 1} & f''(x) & \\
(-\infty, -1) & \text{neg} & \text{neg} & \text{pos} & \text{concave up} \\
(-1, 2) & \text{neg} & \text{pos} & \text{neg} & \text{concave down} \\
(2, \infty) & \text{pos} & \text{pos} & \text{pos} & \text{concave up} \\
\end{array}
\]

-1, 2 are points of inflection.
2. Consider the function

\[ f(x) = \frac{\sqrt{2x^2 - 1}}{x}. \]

(a) Determine the domain of \( f(x) \).

(b) Find the horizontal asymptotes of \( f(x) \), if there are any.

\[(a) \quad 2x^2 - 1 \geq 0 \quad \Rightarrow \quad x^2 \geq \frac{1}{2} \quad \Rightarrow \quad x \geq \frac{1}{\sqrt{2}} \quad \text{and} \quad x \leq -\frac{1}{\sqrt{2}} \]

\[(b) \quad \lim_{x \to \infty} \frac{\sqrt{2x^2 - 1}}{x} = \lim_{x \to \infty} \frac{\sqrt{2x^2 - 1}}{\sqrt{x^2}}
\]

\[= \lim_{x \to \infty} \sqrt{2 - \frac{1}{x^2}} = \sqrt{2 - 0} = \sqrt{2} \]

\[\lim_{x \to -\infty} \frac{\sqrt{2x^2 - 1}}{x} = \lim_{x \to \infty} \frac{\sqrt{2(x^2) - 1}}{-x}
\]

\[= -\lim_{x \to \infty} \frac{\sqrt{2x^2 - 1}}{x} = -\sqrt{2} \]
3. As the Sun sets, the angle of elevation of the Sun above the horizon is decreasing at the rate of \( \frac{1}{4} \) radian/hr. How fast is the shadow cast by a 400-foot-tall building increasing when the angle of elevation of the Sun is \( \frac{\pi}{4} \) radians. Note: the angle of elevation of the Sun is the angle from the ground - assumed flat - up to the Sun.

\[
\theta = \text{angle of elevation} \quad \frac{d\theta}{dt} = -\frac{1}{4}
\]

\[
x = \text{length of shadow}
\]

\[
\tan \theta = \frac{400}{x} = 400x^{-1}
\]

\[
\sec^2 \theta \frac{d\theta}{dt} = -400x^{-2} \frac{dx}{dt}
\]

\[
\sec^2 \left( \frac{\pi}{4} \right) = \frac{1}{\cos^2 \left( \frac{\pi}{4} \right)} = \left( \frac{\sqrt{2}}{2} \right)^2 = (\sqrt{2})^2 = 2
\]

\[
\tan \left( \frac{\pi}{4} \right) = 1 \quad \text{so} \quad x = 400
\]

\[
2 \left( -\frac{1}{4} \right) = -400 \left( 400 \right)^{-2} \frac{dx}{dt}
\]

\[
\frac{dx}{dt} = 200 \text{ ft/hr.}
\]
4. Find all the local maxima and local minima of the function

\[ f(x) = x^{2/3}(1-x)^2. \]

\[
f'(x) = \frac{2}{3} x^{-1/3} (1-x)^2 + x^{2/3} 2(1-x)(-1)
\]

\[
= 2 x^{-1/3} \left( \frac{1}{3} (1-x)^2 - x(1-x) \right)
\]

\[
= 2 x^{-1/3} \left( \frac{1}{3} - \frac{2}{3} x + \frac{1}{3} x^2 - x + x^2 \right)
\]

\[
= 2 x^{-1/3} \left( \frac{4}{3} x^2 - \frac{5}{3} x + \frac{1}{3} \right)
\]

\[
= \frac{2}{3} x^{-1/3} (4x^2 - 5x + 1)
\]

\[
= \frac{2}{3} x^{-1/3} (4x-1)(x-1)
\]

Critical points: \( x = 0, \frac{1}{4}, 1 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-\infty, 0)</th>
<th>( 0, \frac{1}{4} )</th>
<th>( \frac{1}{4}, 1 )</th>
<th>( 1, \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^{-1/3} )</td>
<td>neg</td>
<td>pos</td>
<td>pos</td>
<td>pos</td>
</tr>
<tr>
<td>( 4x-1 )</td>
<td>neg</td>
<td>neg</td>
<td>pos</td>
<td>pos</td>
</tr>
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<td>neg</td>
<td>neg</td>
<td>neg</td>
<td>pos</td>
</tr>
<tr>
<td>( f'(x) )</td>
<td>neg</td>
<td>pos</td>
<td>neg</td>
<td>pos</td>
</tr>
</tbody>
</table>

0, 1 are local min, \( \frac{1}{4} \) local max.
5. Suppose that a function $f(x)$ has second derivative $f''(x)$ for all real numbers $x$.

(a) Prove that there exists a point $c$ in $(0, 1)$ such that

$$f(1) - f(0) = f'(c) (1 - 0) = f'(c).$$

(b) Prove that if $|f''(x)| \leq 1$ for all $x$ in $(0, 1)$, then

$$|f(1) - f(0) - f'(0)| < 1.$$

(a) By MVT

$$f(1) - f(0) = f'(c) (1 - 0) = f'(c).$$

(b) By part (a)

$$f(1) - f(0) - f'(0) = f'(c) - f'(0).$$

By MVT, there exists $d$ in $(0, c)$ with

$$f'(c) - f'(0) = f''(d) (c - 0)$$

Substituting

$$|f(1) - f(0) - f'(0)|$$

$$= |f'(c) - f'(0)| = |f''(d)| |c - 0| < 1$$

Since $|f''(d)| < 1$ and $|c - 0| < 1$. 

Until