1. (a) Differentiate

\[ f(x) = x \sqrt{\sin(x^2 + 1)} \]

(b) Find \( f''(1) \) for

\[ f(x) = \frac{3x}{2x + 1} \]

(a) \( f'(x) = x \left( \sin(x^2 + 1) \right)^{\frac{1}{2}} \)

\( f'(x) = \left( \sin(x^2 + 1) \right)^{\frac{1}{2}} + x \left( \sin(x^2 + 1) \right)^{-\frac{1}{2}} \left( 2x \sin(x^2 + 1) \right) (2x) \)

(5) \( f'(x) = \frac{3(2x+1) - 3x(2)}{(2x+1)^2} = \frac{6x+3-6x}{(2x+1)^2} \)

\[ = \frac{3}{(2x + 1)^2} \]

\( f''(x) = (3) \left( -2 \right) (2x+1)^{-3} (2) \)

\( f''(1) = (3) \left( -2 \right) (3)^{-3} (2) = -\frac{4}{9} \)
2. Find all the values of \( x \) such that the tangent line to the curve
\[
x^2 + y^2 + xy = 1
\]
at \((x, y)\) has slope equal to 2.

\[
2x + 2y \frac{dy}{dx} + y + x \frac{dy}{dx} = 0
\]
\[
\frac{dy}{dx} = - \frac{2x + y}{2y + x}
\]
\[
5y = -4x \quad y = - \frac{4x}{5}
\]
\[
x^2 + \left( - \frac{4}{5} x \right)^2 + x \left( - \frac{4}{5} x \right) = 1
\]
\[
x^2 + \frac{16}{25} x^2 - \frac{4}{5} x^2 = 1
\]
\[
\frac{21}{25} x^2 = 1 \quad x = \pm \sqrt{\frac{25}{21}}
\]
3. Calculate the derivative at \( x = 1 \), using the definition of the derivative, of the function

\[
f(x) = \frac{1}{2x + 1}.
\]

\[
f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{\frac{1}{2x+1} - \frac{1}{3}}{x - 1}
\]

\[
= \lim_{x \to 1} \frac{3 - (2x+1)}{(2x+1)(3)(x-1)} = \lim_{x \to 1} \frac{2 - 2x}{(2x+1)(3)(x-1)}
\]

\[
= \lim_{x \to 1} \frac{2(1-x)}{(2x+1)(3)(x-1)}
\]

\[
= \lim_{x \to 1} \frac{-2}{(2x+1)(3)} = -\frac{2}{9}
\]
4. (a) List the facts about continuous functions stated in the course that you would use to prove that

\[ f(x) = \cos^2 x - \sin x \]

is a continuous function. Do not write a proof, just make a list.

(b) Prove that the equation

\[ \cos^2 x = \sin x \]

has a solution.

(a) sine and cosine are continuous
continuous minus continuous is continuous
continuous times continuous is continuous

(b) \( f(0) = 1 - 0 > 0 \), \( f(\pi/2) = 0 - 1 < 0 \)
f(x) is continuous so \( f(c) = 0 \) for some c in \( (0, \pi/2) \) by the Intermediate Value Theorem.
\[ \cos^2 c - \sin c = 0 \] implies \( \cos^2 c = \sin c \).
5. Evaluate the limits if they exist.

10 points

(a) $\lim_{x \to 0} (\cot x - \csc x) = \lim_{x \to 0} \frac{\cot x - 1}{\csc x} \cdot \frac{1}{x}$

$= \lim_{x \to 0} \frac{\cos x - 1}{\sin x} \cdot \lim_{x \to 0} \frac{1}{x} = 0 \cdot \infty = 0$

(b) $\lim_{x \to 1} \left( \frac{2(\sqrt{x}-1) - (x-1) - (\sqrt{x} - 1)}{(x-1)(\sqrt{x} - 1)} \right) \frac{\sqrt{x} + 1}{\sqrt{x}^2 + 1}$

$= \lim_{x \to 1} \frac{2(x-1) - (x-1) - (\sqrt{x} - 1)}{(x-1)(\sqrt{x} - 1)}$

$= \lim_{x \to 1} \frac{2(x-1) - (x-1) - (\sqrt{x} - 1)}{(x-1)(\sqrt{x} - 1)} = \lim_{x \to 1} \frac{1 - \sqrt{x}}{x-1} \cdot \frac{1 + \sqrt{x}}{1 + \sqrt{x}}$

$= \lim_{x \to 1} \frac{1 - \sqrt{x}}{(x-1)(1+\sqrt{x})} = \lim_{x \to 1} \frac{-1}{1 + \sqrt{x}} = -\frac{1}{2}$