1. (a) Differentiate
\[ f(x) = \sqrt{\sin^3 x + 1} \]
\[ f'(x) = \frac{1}{2} (\sin^3 x + 1)^{-\frac{1}{2}} \frac{d}{dx} (\sin^3 x + 1) \]
\[ = \frac{1}{2} (\sin^3 x + 1)^{-\frac{1}{2}} (3 \sin^2 x)(\cos x) \]

(b) Calculate the second derivative of
\[ f(x) = \frac{x}{x^2 + 1} \]
\[ f'(x) = \frac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} \]
\[ f''(x) = \frac{-2x(x^2 + 1)^2 - (1 - x^2)2(x^2 + 1)(2x)}{(x^2 + 1)^4} \]
2. Find all the values of $x$ for which the tangent line to the curve 

$$xy^2 = y + 3x^3$$

is horizontal.

$$y^2 + 2xy \frac{dy}{dx} = \frac{dx}{dy} + 9x^2$$

$$0 \quad 0$$

$$y^2 = 9x^2 \quad y = \pm 3x$$

$y = 3x$: $x(3x^2) = 3x + 3x^3 \quad 6x^3 - 3x = 0$

$$3x(2x^2 - 1) = 0 \quad x = 0, x = \pm \sqrt{\frac{1}{2}}$$

$y = -3x$: $9x^3 = -3x + 3x^2 \quad 6x^3 + 3x = 0$

$$3x(2x^2 + 1) = 0 \quad x = 0$$
3. Prove that \[ \frac{\cos x}{\sin x + 2} = x \]
for some \( x \) in \( [0, \pi/2] \).

Let \( f(x) = x - \frac{\cos x}{\sin x + 2} \), which is continuous for all \( x \) since \( \sin x + 2 \neq 0 \).

\[ f(0) = 0 - \frac{1}{0+2} = -\frac{1}{2} < 0 \]

\[ f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{0}{1+2} = \frac{\pi}{2} > 0 \]

so \( f(x) = 0 \) for some \( x \) in \( [0, \pi/2] \) by IVT and for that \( x \)

\[ \frac{\cos x}{\sin x + 2} = x \]
4. Find the value of \( a \) for which the limit
\[
\lim_{x \to 2} \frac{x^2 - \frac{5}{3}x + a}{x - 2}
\]
exists and calculate the limit.

The limit can exist only if
\[
\lim_{x \to 2} \left( x^2 - \frac{5}{3}x + a \right) = 0
\]
\[
\lim_{x \to 2} \left( x^2 - \frac{5}{3}x + a \right) = 4 - \frac{5}{3}(2) + a = \frac{3}{3} + a
\]

so \( a = -\frac{2}{3} \)

\[
\lim_{x \to 2} \frac{x^2 - \frac{5}{3}x - \frac{2}{3}}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + \frac{1}{2})}{x - 2} = 2 + \frac{1}{3}
\]
5. (a) Write the definition of the derivative \( g'(x) \) of a function \( g(x) \) that is the limit of a quotient in which the denominator is \( h \).

(b) Use the definition of part (a) to prove that if \( g(x) = xf(x) \), then

\[
g'(x) = xf'(x) + f(x).
\]

(Do not use Leibniz’ Rule.)

\[
(a) \quad g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{(x+h)f(x+h) - xf(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{xf(x+h) + hf(x+h) - xf(x)}{h}
\]

\[
= \lim_{h \to 0} x \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{hf(x+h)}{h}
\]

\[
= xf'(x) + f(x)
\]