1. Find the (absolute) maximum and minimum of

\[ f(x) = \frac{\sin x + \cos x}{2} \]

on the interval \([0, \pi]\), showing your work.

\[ f'(x) = \frac{1}{2} (\cos 2x - \sin x) = 0 \]

\[ \cos 2x = \sin x \quad \Rightarrow \quad 2x = \frac{\pi}{4} \]

\[ f\left(\frac{\pi}{4}\right) = \frac{\sin \left(\frac{\pi}{4}\right) + \cos \left(\frac{\pi}{4}\right)}{2} = \frac{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}}{2} = \frac{\sqrt{2}}{2} \max \]

\[ f(0) = \frac{0 + 1}{2} = \frac{1}{2} \]

\[ f(\pi) = \frac{0 + (-1)}{2} = -\frac{1}{2} \min \]
2. Use the Mean Value Theorem to prove that if \( f(1) = 3 \) and \( f'(x) < -1 \) for all \( x \geq 0 \), then \( f(4) \) is negative.

By the MVT

\[
\frac{f(4) - f(1)}{4 - 1} = f'(c) < -1
\]

for some \( c \) in \([1, 4]\)

\[
\frac{f(4) - 3}{3} < -1 \Rightarrow f(4) - 3 < -3 \Rightarrow f(4) < 0
\]
3. A function is defined by

\[ f(x) = \frac{x}{\sqrt{2x^2 + 1}}. \]

(a) Find the horizontal asymptotes of \( f(x) \), if it has any.

(b) Given that

\[ f'(x) = \frac{1}{\sqrt{(2x^2 + 1)^3}}, \]

determine the intervals on which \( f(x) \) is concave up and the intervals on which \( f(x) \) is concave down.

\[
(\text{a}) \quad \lim_{x \to \infty} \frac{x}{\sqrt{2x^2 + 1}} \cdot \frac{1}{x} = \lim_{x \to \infty} \frac{1}{\sqrt{2 + \frac{1}{x^2}}} = \frac{1}{\sqrt{2}}.
\]

\[
\lim_{x \to -\infty} \frac{x}{\sqrt{2x^2 + 1}} = \lim_{x \to \infty} \frac{-x}{\sqrt{2(-x)^2 + 1}} = -\lim_{x \to \infty} \frac{x}{\sqrt{2x^2 + 1}} = -\frac{1}{\sqrt{2}}.
\]

(\text{b}) \quad f'(x) = (2x^2 + 1)^{-3/2}

\[
f''(x) = -\frac{3}{2} (2x^2 + 1)^{-5/2} (4x) = \frac{-6x}{(2x^2 + 1)^{5/2}}.
\]

Concave up on \((-\infty, 0)\)

Concave down on \((0, \infty)\)
4. Given that \( f''(x) = \sin x + 6x, \)
\( f'(\pi) = 1 \) and \( f(\pi) = 0, \) find \( f(x). \)

\[
f'(x) = \int \sin x + 6x \, dx \]
\[
= -\cos x + 3x^2 + C
\]
\[
f'(\pi) = -\cos (\pi) + 3\pi^2 + C = 1
\]
\[
-(-1) + 3\pi^2 + C = 1 \quad \Rightarrow \quad C = -3\pi^2
\]

\[
f(x) = \int -\cos x + 3x^2 - 3\pi^2 \, dx \]
\[
= -\sin x + x^3 - 3\pi^2 x + D
\]
\[
f(\pi) = -\sin (\pi) + \pi^3 - 3\pi^2 (\pi) + D = 0
\]
\[
0 + \pi^3 - 3\pi^3 + D = 0 \quad \Rightarrow \quad D = 2\pi^3
\]

\[
f(x) = -\sin x + x^3 - 3\pi^2 x + 2\pi^3
\]
5. A kite 100 feet above the ground moves horizontally at a rate of 8 feet per second. At what rate is the angle \( \theta \) between the kite string and the (horizontal) ground changing when the length of the string is 200 feet? (Note: Think of the kite string as a straight line from a point on the ground to the kite.)

\[
\begin{align*}
100 & \quad \frac{dx}{dt} = 8 \quad \frac{d\theta}{dt} = ? \\
\tan \theta & = \frac{100}{x} = 100x^{-1} \\
\sec^2 \theta \frac{d\theta}{dt} & = -100x^{-2} \frac{dx}{dt} \\
\sec^2 \theta \frac{d\theta}{dt} & = -\frac{800}{x^2} \\
\frac{d\theta}{dt} & = -\frac{800}{x^2} \cos^2 \theta \\
\end{align*}
\]

If \( \frac{200}{\sqrt{100}} \), then \( \cos \theta = \frac{x}{200} \)

\[
\frac{d\theta}{dt} = -\frac{800}{x^2} \left( \frac{x}{200} \right)^2 = -\frac{800}{(200)^2} = -\frac{1}{50}
\]