1. (a) Differentiate

\[ f(x) = \sqrt{\sin^3 x + 1} \]

(b) Calculate the second derivative of

\[ f(x) = \frac{x}{x^2 + 1} \]

(a) \[ f(x) = (\sin^3 x + 1)^{\frac{1}{2}} \]

\[ f'(x) = \frac{1}{2} (\sin^3 x + 1)^{-\frac{1}{2}} \frac{d}{dx} (\sin^3 x + 1) \]

\[ = \frac{1}{2} (\sin^3 x + 1)^{-\frac{1}{2}} (3 \sin^2 x) (\cos x) \]

(b) \[ f'(x) = \frac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} \]

\[ f''(x) = \frac{-2x(x^2 + 1)^2 - (1 - x^2)2(x^2 + 1)(2x)}{(x^2 + 1)^4} \]
2. Find all the values of $x$ for which the tangent line to the curve

$$xy^2 = y + 3x^3$$

is horizontal.

$$y^2 + 2xy \frac{dy}{dx} = \frac{dy}{dx} + 9x^2$$

$$\frac{dy}{dx} = 0 \quad \frac{dy}{dx} = 0$$

$$y^2 = 9x^2 \quad y = \pm 3x$$

$y = 3x$: $x(3x^2) = 3x + 3x^3 \quad 6x^3 - 3x = 0$

$$3x(2x^2 - 1) = 0 \quad x = 0, x = \pm \sqrt{\frac{1}{2}}$$

$y = -3x$: $9x^3 = -3x + 3x^2 \quad 6x^3 + 3x = 0$

$$3x(2x^2 + 1) = 0 \quad x = 0$$
3. Prove that \( \frac{\cos x}{\sin x + 2} = x \) for some \( x \) in \([0, \pi/2]\).

Let \( f(x) = x - \frac{\cos x}{\sin x + 2} \) which is continuous for all \( x \) since \( \sin x + 2 \neq 0 \).

\[ f(0) = 0 - \frac{1}{0 + 2} = -\frac{1}{2} < 0 \]

\[ f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{0}{1 + 2} = \frac{\pi}{2} > 0 \]

So \( f(x) = 0 \) for some \( x \) in \([0, \pi/2]\) by IVT and for that \( x \)

\[ \frac{\cos x}{\sin x + 2} = x \]
4. Find the value of $a$ for which the limit
\[ \lim_{x \to 2} \frac{x^2 - \frac{5}{3}x + a}{x - 2} \]
exists and calculate the limit.

The limit can exist only if
\[ \lim_{x \to a} x^2 - \frac{5}{3}x + a = 0 \]

\[ \lim_{x \to a} x^2 - \frac{5}{3}x + a = 4 - \frac{5}{3}(a) + a = \frac{2}{3} + a \]

So $a = -\frac{2}{3}$

\[ \lim_{x \to a} \frac{x^2 - \frac{5}{3}x - \frac{2}{3}}{x - 2} \]

\[ = \lim_{x \to a} \frac{(x-a)(x+\frac{1}{3})}{x-a} = 2 + \frac{1}{3} \]
5. (a) Write the definition of the derivative \( g'(x) \) of a function \( g(x) \) that is the limit of a quotient in which the denominator is \( h \).

\( \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \)

(b) Use the definition of part (a) to prove that if \( g(x) = xf(x) \), then

\[ g'(x) = xf'(x) + f(x). \]

(Do not use Leibniz' Rule.)

\[
\begin{align*}
(a) \quad g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)f(x+h) - xf(x)}{h} \\
&= \lim_{h \to 0} \frac{xf(x+h) + hf(x+h) - xf(x)}{h} \\
&= \lim_{h \to 0} \frac{xf(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{hf(x+h)}{h} \\
&= xf'(x) + f(x)
\end{align*}
\]