Today we’ll be interested in various properties of quadratic (degree two) polynomials. The standard form of such a polynomial is

\[ ax^2 + bx + c \]  

where \( a, b, \) and \( c \) are numbers and \( x \) is a variable. There isn’t much we can say about our quadratic just by looking at it in this form, so our goal will be to use algebraic manipulations to write

\[ ax^2 + bx + c = a(x - d)^2 + e \]  
\[ ax^2 + bx + c = a(x - r_1)(x - r_2) \]

for some numbers \( d \) and \( e \) and some other (possibly complex) numbers \( r_1 \) and \( r_2 \). First, let’s start with a warmup:

**Problem 1** Prove the following (extremely important) formulas:

\[(a + b)(c + d) = ac + ad + bc + bd\]

\[(a + b)(a - b) = a^2 - b^2\]
\[(a + b)^2 = a^2 + 2ab + b^2\]

**Problem 2** Prove one more important formula:

\[(x - a)(x - b) = x^2 - (a + b)x + ab\]

We can use problem 2 to put some especially simple quadratics into the form (3) from page 1. For instance, if our quadratic is

\[x^2 - 4x + 3,\]

if we want to write \(x^2 - 4x + 3 = (x - r_1)(x - r_2),\) then by problem 2, all we have to do is find two numbers \(r_1\) and \(r_2\) that add up to 4 and multiply to 3, like 1 and 3! Now it’s easy to see that \(x^2 - 4x + 3 = (x - 1)(x - 3).\) Also, it’s very easy to solve the equation \(x^2 - 4x + 3 = 0\ now: there are exactly two solutions, \(x = 1\) and \(x = 3.\)

**Problem 3** Factor the following quadratics:

\[x^2 - 8x + 16\]

\[x^2 - 7x + 10\]

\[x^2 + x - 2\]
**Problem 4** Solve the following equations:

Hint: \( ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \)

\[ 3x^2 - 3 = 0 \]

\[ 4x^2 - 8x - 32 = 0 \]

\[ 2x^2 + 4x + 2 = 0 \]

So it’s pretty easy to solve a quadratic if it happens to have integer roots, but unfortunately, most “real life” quadratics do not. For these, we will use a method called completing the square to convert our quadratic to form (2) from page 1. Here’s the idea:

Suppose we start with the quadratic

\[ x^2 - 4x + 1, \]

and we want to write it in the form

\[ x^2 - 4x + 1 = (x - d)^2 + e. \]

Note that if we expand this equation, we get

\[ x^2 - 4x + 1 = (x - d)^2 + e = x^2 - 2dx + d^2 + e. \]

From this we see that \( 2d = 4 \) or \( d = 2 \). So,
\[ x^2 - 4x + 1 = (x - 2)^2 + e = x^2 - 4x + 4 + e. \]

But now if we set \( e = -3 \), then \( 4 + e = 1 \), so

\[ (x - 2)^2 - 3 = x^2 - 4x + 1, \]

as desired.

**Problem 5** Express the following quadratics in form (2):

\[ x^2 + 2x - 3 \]

\[ x^2 - 8x + 26 \]

**Problem 6** Express the following quadratic in form (2), where \( b \) and \( c \) are arbitrary numbers:

\[ x^2 + bx + c \]

**Problem 7** Express the following quadratic in form (2), where \( a, b, \) and \( c \) are arbitrary numbers and \( a \neq 0 \):

\[ ax^2 + bx + c \]
**Problem 8** Form (2) is very nice, partly because it makes it easy to graph our quadratic, and to find their minimum or maximum value. Sketch a graph of each of the following quadratics. Be sure to include in your sketch the location and value of the maximum or minimum of the function.

\[ x^2 + 2x - 3 \]

\[ -2x^2 + 8x - 2 \]

**Problem 9** A dog owner has 100 feet of fence and wants to fence off a rectangular region in his yard. What is the largest area he can possibly fence off?
**Problem 10** Solve the following quadratics by first putting them in form (2):

\[ x^2 + 2x - 3 \]

\[ -2x^2 + 8x - 2 \]

**Problem 11** Solve the general quadratic equation

\[ ax^2 + bx + c = 0 \]

using your solution to problem 7.

You should have found the two solutions

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \]

This is called the quadratic formula, and while most times in math it’s better to reason things out, this formula is **worth memorizing.**
Problem 12 A ball is thrown into the air by a $y_0 = 2$ m tall person with initial speed $v_0 = 20$ m/s. The height $y$ of the ball is known to satisfy

$$y = y_0 + v_0 t + \frac{1}{2}gt^2$$

where $g = 9.8 \text{ m/s}^2$ is the acceleration of gravity and $t$ is the elapsed time in seconds. At what time does the ball hit the ground?

Problem 13 Given a quadratic $ax^2 + bx + c$, define the discriminant of to be $\Delta = b^2 - 4ac$. Use the quadratic formula to prove the following about the equation $ax^2 + bx + c = 0$:

The equation has exactly two real solutions if and only if $\Delta > 0$.

The equation has exactly one real solution if and only if $\Delta = 0$.

The equation has no real solution if and only if $\Delta < 0$. 
Problem 14 Prove that if \( ax^2 + bx + c \geq 0 \) for every real number \( x \). Prove that \( \Delta \leq 0 \) (a picture and half a sentence would suffice).

Problem 15 Apply the result from problem 14 to the quadratic:

\[
(a_1 - xb_1)^2 + (a_2 - xb_2)^2 + \cdots + (a_n - xb_n)^2.
\]

Problem 16 By the quadratic formula, the equation

\[ x^2 + bx + c = 0 \]

has solutions

\[
r_1 = \frac{-b + \sqrt{b^2 - 4c}}{2}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4c}}{2}.
\]

Prove that \( r_1 + r_2 = -b \) and \( r_1r_2 = c \).