Warm Up

Find the product of the following permutations by first writing the permutations in their expanded form and then calculating their products.

\[(1) \; (3 \; 1 \; 4 \; 2) \circ (3 \; 2 \; 1 \; 4)\]

Remember that the permutation to the right is the one that is applied first!
(2) \( (4 \ 1 \ 3 \ 2) \circ (2 \ 3 \ 4 \ 1) \)

(3) \( (2 \ 5 \ 1 \ 3 \ 4) \circ (3 \ 4 \ 1 \ 5 \ 2) \)
Further Improving Notation

Let us take another look at the permutation

$$\sigma = \begin{pmatrix} 3 & 2 & 4 & 1 \end{pmatrix}$$

The permutation does not shuffle the second element. Hence, writing it is redundant. Knowing that the original set consists of four elements, we can write the permutation down as

$$\sigma = \begin{pmatrix} 3 & 4 & 1 \end{pmatrix}$$

This convention becomes very convenient with larger permutations.

**Problem 1.** In Sam Loyd’s problem with the 15 puzzle, we were given a 15 puzzle with the following orientation.

```
1  2  3  4
5  6  7  8
9 10 11 12
13 15 14
```

This is represented by the permutation

$$\sigma = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 15 \ 14 \ 16)$$

The permutation only switches the 14th and 15th elements only. What is this permutation written in the new notation?

$$\sigma = \begin{pmatrix} 13 & 15 \ 14 \end{pmatrix}$$

**Problem 2.** The permutation $$\nu = \begin{pmatrix} 3 & 1 \end{pmatrix}$$ acts on a set of four elements. Write down its full version.

$$\nu = \begin{pmatrix} 2 & 3 & 4 \ 2 & 4 \end{pmatrix}$$
Transpositions

A permutation that swaps two elements and doesn’t shuffle anything else is called a *transposition*. For example, the permutation that switches the order of the third and fifth element in a six-element set is

\[
(5 \ 3) = (1 \ 2 \ 5 \ 4 \ 3 \ 6)
\]

**Problem 3.** What is the inverse of the transposition \((5 \ 3)\)?

\[
(5 \ 3)^{-1} = (5 \ 3)
\]

Permutations as a Product of Transpositions

Any permutation can be re-written as a product of transpositions. For example, let us consider the permutation \(\sigma = (3 \ 1 \ 4 \ 2)\).

Applying the transposition \((3 \ 1)\) to the original order of the elements gives us the following

\[
(1 \ 2 \ 3 \ 4) \xrightarrow{(3 \ 1)} (3 \ 2 \ 1 \ 4)
\]

Then the transposition \((3 \ 2)\) to the result gives us the following

\[
(3 \ 2 \ 1 \ 4) \xrightarrow{(3 \ 2)} (3 \ 1 \ 2 \ 4)
\]

Finally, applying the transposition \((4 \ 3)\) to the result gives us the permutation \(\sigma = (3 \ 1 \ 4 \ 2)\)

\[
(3 \ 1 \ 2 \ 4) \xrightarrow{(4 \ 3)} (3 \ 1 \ 4 \ 2)
\]

Instead of applying \((3 \ 1 \ 4 \ 2)\), we can apply \((3 \ 1)\), \((3 \ 2)\) and then \((4 \ 2)\).

So we can say that

\[
(3 \ 1 \ 4 \ 2) = (4 \ 3) \circ (3 \ 2) \circ (3 \ 1)
\]
Problem 4. Write the product of the following transpositions for a 5-element set. Remember to apply the permutation on the right first!

(1) \((5 \ 4) \circ (3 \ 1) \circ (3 \ 2)\)

\[
\begin{align*}
(1 & \ 2 & 3 & 4 & 5) \\
\overset{(3 \ 1)}{\rightarrow} & (2 & 3 & 4 & 5) \\
\overset{(5 \ 4)}{\rightarrow} & (2 & 3 & 1 & 4 & 5)
\end{align*}
\]

(2) \((2 \ 1) \circ (4 \ 2) \circ (5 \ 3)\)

\[
\begin{align*}
(1 & \ 2 & 3 & 4 & 5) \\
\overset{(5 \ 3)}{\rightarrow} & (1 & 2 & 5 & 4 & 3) \\
\overset{(4 \ 2)}{\rightarrow} & (1 & 4 & 5 & 2 & 3)
\end{align*}
\]

(3) \((3 \ 2) \circ (3 \ 2) \circ (4 \ 1) \circ (4 \ 1) \circ (5 \ 2) \circ (5 \ 2) \circ (3 \ 1) \circ (3 \ 1) \circ (4 \ 3) \circ (4 \ 3)\)

Hint: You can simplify this. Look at Problem 3.

\[
(3 \ 2) \circ (3 \ 2) = \text{e}
\]

So every permutation is undone by the next, so everything is undone in the end.

So the final permutation is just \(\text{e}\).
Problem 5. Re-write the permutation \( (2 \ 3 \ 1) \) as a product of transpositions.

\[
\begin{align*}
(1 \ 2 \ 3) & \xrightarrow{(2 \ 1)} (2 \ 1 \ 3) \\
(2 \ 1 \ 3) & \xrightarrow{(3 \ 2)} (2 \ 3 \ 1)
\end{align*}
\]

So \( (2 \ 3 \ 1) = (3 \ 2) \circ (2 \ 1) \)

Problem 6. Re-write the permutation \( (4 \ 3 \ 1 \ 2) \) as a product of transpositions.

\[
\begin{align*}
(1 \ 2 \ 3 \ 4) & \xrightarrow{(4 \ 1)} (4 \ 2 \ 3 \ 1) \xrightarrow{(3 \ 2)} (4 \ 3 \ 2 \ 1) \\
& \xrightarrow{(4 \ 3)} (4 \ 3 \ 1 \ 2)
\end{align*}
\]

So \( (4 \ 3 \ 1 \ 2) = (4 \ 3) \ast (3 \ 2) \ast (4 \ 1) \)
Problem 7. Re-write the permutation \( (3 \ 4 \ 1 \ 2) \) as a product of transpositions.

\[
\begin{align*}
(1 \ 2 \ 3 \ 4) & \xrightarrow{(3 \ 1)} (3 \ 2 \ 1 \ 4) \xrightarrow{(4 \ 2)} (3 \ 4 \ 1 \ 2) \\
(1 \ 2 \ 3 \ 4 \ 5) & \xrightarrow{(3 \ 1)} (3 \ 2 \ 1 \ 4 \ 5) \xrightarrow{(5 \ 2)} (3 \ 5 \ 1 \ 4 \ 2) \xrightarrow{(5 \ 4)} (3 \ 5 \ 1 \ 2 \ 4)
\end{align*}
\]
Problem 9. We will now see how permutations can represent the moves made on the 15 puzzle.

(1) Write down the permutation $\sigma_1$ that corresponds to the orientation of the puzzle on the left. Remember, we treat the empty square as the 16th tile!

$$\sigma_1 = \{15 \mid 4\}$$

(2) Write down the permutation $\sigma_2$ that corresponds to the orientation of the puzzle on the right as a single permutation.

$$\sigma_2 = \{16 \mid 15 \mid 14 \mid 12\}$$

(3) Write down the permutation $\mu$ that corresponds to the move shown above of the 15 puzzle.

$$\mu = \{16 \mid 12\}$$

(4) Apply $\mu$ to $\sigma_1$. That is, calculate $\mu \circ \sigma_1$.

$$\mu \circ \sigma_1 = \{16 \mid 15 \mid 14 \mid 12\}$$

(5) Notice that $\mu \circ \sigma_1 = \sigma_2$. Why does this make sense?

They do equal each other. This makes sense because applying $\mu$ to $\sigma_1$ is the same as switching the 12 and 16th tile, which would give us the orientation shown as $\sigma_2$. 
Parity of a Permutation

An inversion of a permutation occurs when a smaller number is moved to the right of a larger number.

For example, the permutation $\sigma = (5\ 1\ 4\ 3\ 2)$ moves 5 to the first position so $(5\ 1)$, $(5\ 2)$, $(5\ 3)$, $(5\ 4)$ are all inversions of $\sigma$.

Note that although the words “inverse” and “inversion” are very similar, the inverse of a permutation and the inversions of a permutation are very different!

**Problem 10.** Write down all other inversions of the permutation $\sigma = (5\ 1\ 4\ 3\ 2)$.

$(5\ 1)$ $(5\ 4)$ $(5\ 3)$ $(5\ 2)$ $(4\ 3)$ $(4\ 2)$ $(3\ 2)$

**Problem 11.** Write down all inversions of the permutation $\sigma = (3\ 2\ 1)$.

$(3\ 2)$ $(3\ 1)$ $(2\ 1)$
The sign of a permutation is defined according to the following formula:

\[ sgn(\sigma) = (-1)^{N(\sigma)} \]

where \( N(\sigma) \) is the number of versions of the permutation \( \sigma \).

For example, the total number of inversions in \( \sigma = (3 \ 2 \ 1) \) is 2, so \( sgn(\sigma) = (-1)^2 = 1 \).

**Problem 12.** What is the sign of the trivial permutation?

\[ sgn(e) = (-1)^0 = 1 \]

**Problem 13.** Find the signs of the following permutations

1. \( sgn(3 \ 1 \ 4 \ 2) \)
   - There are 3 inversions, so
   \[ sgn(3 \ 1 \ 4 \ 2) = (-1)^3 = -1 \]

2. \( sgn(3 \ 2 \ 4 \ 1) \)
   - There are 4 inversions, so
   \[ sgn(3 \ 2 \ 4 \ 1) = (-1)^4 = 1 \]

**Problem 14.** What is the sign of the permutation corresponding to the following configuration of the 15 puzzle? (Remember, the empty square is considered the 16th tile.)

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 \color{gray}{11} & \color{gray}{12} \\
13 & 15 & 14 & 12 \\
\end{array}
\]

This orientation of the 15 puzzle corresponds to \( (16 \ 11 \ 15 \ 14 \ 12) \) so there are 7 inversions. Thus, the sign is \((-1)^7 = -1\).

Another way we can do this is count the tiles to the right of each tile that has a value less than the first tile. There are 9, which gives \((-1)^9\). We will later see why
**Taxicab Geometry**

We are not pretty close to figuring out whether or not a solution exists for an orientation of the 15 puzzle, but we still need to learn some taxicab geometry before tying everything together.

Imagine that you take a taxicab to get from point A to point B in a city with streets and avenues forming a rectangular pattern.

![Taxicab diagram](image)

Similar to Euclidean geometry, there exists a shortest path. Unlike Euclidean geometry, the shortest path is not unique.

**Problem 15.** On the picture above, draw two different shortest paths from A to B.

The point A lies at the intersection of the 1st Ave. and the 4th St. Let us write this fact down as follows

\[ A = (1, 4) \]

B lies at the intersection of the 5th Ave. and the 2nd St.

\[ B = (5, 2) \]

Let \( a \) be the distance between two neighboring avenues and let \( s \) be the distance between two neighboring streets. No matter what shortest path the cab driver chooses, he needs to drive 4 blocks East and 2 blocks South. We can write this as

\[ d_{tc}(A, B) = 4a + 2s \]

**Problem 16.** Find the Euclidean distance \( d_E(A, B) \) between the points A and B.

\[ d_E(A, B) = \sqrt{(4a)^2 + (2s)^2} \]
Problem 17. Without doing any computations, put the correct sign, ≥, ≤, or = between the distance below. Explain your choice.

\[ d_E(A, B) \quad \leq \quad d_{tc}(A, B) \]

Problem 18. For the grid below, \( a = s = 1 \). Find the following taxicab distances.

\[ d_{tc}(A, B) = 7 \]

\[ d_{tc}(A, C) = 12 \]

\[ d_{tc}(B, C) = 5 \]
Problem 19. On the grid below, mark all the points that have the taxicab distance 6 from the point O using a pencil. Next, mark all the points that have a Euclidean distance 6 from the point O using a pen.

\[
\text{(a circle of radius 6)}
\]

Problem 20. Find the taxicab distance from the current position of the empty square to the lower-right corner of the 15 puzzle.
Math Kangaroo

(1) The five-digit natural number \(2 \, 4 \, x \, 8 \, y\) is divisible by 4, 5, and 9. What is the sum of the digits \(x\) and \(y\)?

Since the number must be divisible by 5, the last digit must be a 5 or a 0.
Furthermore, the number must be divisible by 4, so the number must be an even number, so the last digit must be even as well.

The two facts above means that \(y\) must be 0.
For a number to be divisible by 9, the sum of its digits must be divisible by 9.
So \(2 + 4 + x + 8 + y = 2 + 4 + 8 + x = 14 + x\). The sum must be 18, so \(x = 5\).
Finally, \(2 + 4 + 5 + 8 + 0 = 18\)

(2) Alex lights a candle every ten minutes. Each candle burns for 40 minutes and then goes out. How many candles are alight 55 minutes after Alex lit the first candle?

<table>
<thead>
<tr>
<th>Time, lit</th>
<th>0 minutes</th>
<th>10 minutes</th>
<th>20 minutes</th>
<th>40 minutes</th>
<th>50 minutes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time passed after 55th min</td>
<td>55-0 &gt; 40</td>
<td>55-10 &gt; 40</td>
<td>55-20 &gt; 40</td>
<td>55-40 &lt; 40</td>
<td>55-50 &lt; 40</td>
</tr>
</tbody>
</table>

So the first two go out as they have burned for longer than 40 minutes by the 55 minutes mark. This leaves 4 candles.

(3) A sequence starts with 1, -1, -1, 1, -1. After the fifth term, every term is equal to the product of the two preceding terms. For example, the sixth term is equal to the product of the fourth term and the fifth term. What is the sum of the first 2013 terms?

If we write out some terms of the sequence, we have:

\[1, -1, 1, -1, -1, 1, -1, -1, 1, \ldots\]

Notice how they repeat in cycles of three.

The sum of each cycle is -1, and there are 671 cycles (because \(2013 \div 3 = 671\)).
So the sum of the first 2013 digits of the sequence is given by \(671 \times (-1) = -671\).