First Idea: think of a crossing change as of a continuous process. Critical moment — when 2 strands intersect. At this moment, the object we have is not an ordinary knot.

Singular knots: Extend the class of objects we consider to include singular knots, i.e., smooth (continuous) maps \(k: S^1 \rightarrow \mathbb{R}^3\) with no singularities except a finite number of transversal self-intersections (double pts of \(k\)).

Relation between invariants of ordinary and singular knots:

\[
V^{(0)} = V
\]

\(V^{(m)} - \text{invariant of singular knots with exactly } m \text{ double points.}

\[
V^{(m)}(X) = V^{(m-1)}(\nearrow) - V^{(m-1)}(\searrow)
\]

double pt (we will circle them)

Analogy with derivatives in calculus:

\(V^{(m)} \sim m^{th} \text{ partial derivative (of a knot invariant } V)\)

\(V^{(m-1)} \sim (m-1)^{th} \text{ partial derivative}\)

The formula above is: derivatives as "differences."
if $V^{(m+1)} = 0$ (for all singular knots with 
$(m+1)$ double pts).

(The main relation then implies that $V^{(n)} = 0$
(This is recorded as $V \underbrace{(X X \ldots X)}_{m+1} = 0 \quad \forall n \geq m+1$).

**Remark.** Assuming Vassiliev invariants exist,
they don't have to be unique
(e.g., can multiply

**Questions:**
1. Which known knot invariants are Vassiliev invariants?
2. Do Vassiliev invariants classify knots?
3. Is there an analog of Taylor's formula
(i.e., can an arbitrary knot invariant be approximated by Vassiliev invariants?)

**Simple properties.**

**Lemma 1.** $V (\lambda) = 0$ (invariance).

**Proof.**

$V (\lambda) = V(\lambda) - V(\emptyset) = 0,$

since the two knots on the right hand side are equiv.

More generally, for any knot of the form

$k = \begin{array}{c}
\text{A} \\
\text{B} \\
\end{array}$, $V(k) = 0$. (1-term relation)
Lemma 2.

\[ V(\rightarrow) - V(\rightarrow) + V(\rightarrow) - V(\rightarrow) = 0 \]

Proof. Use the double points indicate above in the main relation of Vassiliev invariants. (Notice that we always keep the same double pt untouched). Then we get the sum of the form

\[(a-b) - (c-d) + (c-a) - (d-b) = 0.\]

Vector space structure

\[ V^m - \text{inv. of order } m, \quad V = V V^m \]

\[ V^0 \subset V^1 \subset V^2 \subset V^3 \subset \ldots \]

If \( v, w \in V \), then \( u = \alpha v + \beta w \) satisfies the same linear relation.

Thus, \( V \) is a vector space (inf.-dim.).

It is filtered by the sequence of inclusions

\[ V_0 \subset V_1 \subset V_2 \subset \ldots \]

Invariants of order 0

\( v_0 \in V^0 \) vanishes on any knot with 1 double pt.

Thus,

\[ v_0(\lambda^3) = v_0(\lambda^3) - v_0(\lambda^3) = 0. \]

In particular, \( v_0 \) is invariant under a crossing change.

Since by a series of crossing changes any knot can be modified to become the unknot, \( v_0 \) has the same value on all ordinary knots.
Lemma. \( v_1 \) vanishes on a singular knot with just 1 double pt.

Proof. Let \( K_1 \) be a knot with one double pt. Let \( K'_1 \) be obtained from \( K \) by 1 crossing change.

Then \( v_1(K) - v_1(K'_1) = v_1(K_2) = 0 \)

where \( K_2 \) has 2 double pts.

Thus, \( v_1(K_1) = v_1(K'_1) \).

Thus, \( v_1 \) has the same value on all the knots in \( V^1 \).

By crossing changes, any knot in \( V^1 \) can be modified to \( \infty \).

All Vassiliev invariants on this knot vanish by 1-term relation. Thus, \( V^1 = V^0 = 1R \).

More generally,

Lemma. The value of an invariant \( v_n \) of order \( n \) on a knot with exactly \( n \) double pts does not vary under crossing changes in the knot.

Proof. Let \( K_n \) be of order \( n \) (has \( n \) double pts).

\[ v_n(K_n) - v_n(K_n') = v_n(K_{n+1}) = 0 \]

Thus, \( v_n(K_n) \) is the same as \( v_n(\underbrace{\infty \cdots \infty}_{n \text{ double pts}}) \).
Let \( K: S^1 \to \mathbb{R}^3 \) be a knot with \( n \) double pts.

**Moving around a circle**

**Gauss diagram:**
- oriented circle;
- marked finitely many chords, (up to orientation-preserving diffeos)

**Ex.:**

\[
\begin{align*}
&1=4 \\
&2=5 \\
&3=6
\end{align*}
\]

Gauss diagrams of different orders: (order = \( \# \) (double pts))

\[
\begin{align*}
\text{n=1:} & \quad \circle & \quad \text{n=2:} & \quad \text{&} & \quad \text{n=3:} & \quad \text{&} & \quad \text{&}
\end{align*}
\]

**Lemma:** Any Gauss diagram is a diagram of some singular knot.

\[
\begin{align*}
&1=4 \\
&2=3
\end{align*}
\]
Symbol of $V_n$ = Restriction of $V_n$ to knots with exactly $n$ double pts.

Lemma.
The value of $V_n$ on a diagram knot with $n$ double pts depends only on the Gauss diagram of the knot.

One-term relations on the language of diagrams:

$\bigcirc = 0$ ; $\bigcirc \cdots \bigcirc = 0.$

4-term relation:

$\bigotimes - \bigodot + \bigodot - \bigodot = 0.$

E.g., let $D = \begin{array}{c}
\bigotimes \\
\bigodot \\
\bigodot
\end{array}$ be a Gauss diagram.

More generally,

Thm. Vassiliev invariants of type $m$ (i.e., $V_m$) $\iff$ Functions on Gauss diagrams with $2m$ points satisfying properties analogous to 1-term and 4-term relations.
the function on such a diagram is 0: \( \odot = 0 \)

2. (see above, \( 4 \)-term relation).

\begin{center}
\textbf{Examples.}
\end{center}

\( \boxed{1} \). Coefficients of the Conway polynomial.

Recall the skein relation:

\[
C(\downarrow\downarrow) - C(\downarrow\uparrow) = z \cdot C(\uparrow\uparrow)
\]

Then

\[
C(\uparrow\downarrow) = C(\downarrow\downarrow) - C(\downarrow\uparrow) = z \cdot C(\uparrow\uparrow)
\]

If \( K \) has \( 1 \) double pt, \( C(K) \) is divisible by \( z \).

Inductively, if \( K \) has \( (m+1) \) double points, \( C(K) \) is divisible by \( z^{m+1} \). Thus, the coefficient of \( z^m \) in \( C(K) \) is 0. Thus, \((m+1)\text{th}\) coefficient of \( C(K) \) is a Vassiliev invariant of type \((m+1)\).

\( \boxed{2} \). Coefficients of the \textit{HOMFLY} polynomial.

Skein relation:

\[
a \cdot P(\downarrow\downarrow) - a^{-1} \cdot P(\downarrow\uparrow) = z \cdot P(\uparrow\uparrow)
\]

Another parameterization:

\[
z = \sqrt{q} - \frac{1}{\sqrt{q}}, \quad a = q^{N/2}.
\]

Get

\[
q^{-N/2} \cdot P(\downarrow\downarrow) - q^{N/2} \cdot P(\downarrow\uparrow) = (\sqrt{q} - \frac{1}{\sqrt{q}}) \cdot P(\uparrow\uparrow).
\]

Substitute \( q = e^z \) and expand in powers of \( z \).

The above equation:

\[
P(\downarrow\downarrow) - P(\downarrow\uparrow) = z \cdot (\text{something}).
\]

Similarly to ex.\( \boxed{1} \), \( m \text{th} \) coefficient is inv. of type \( m \).
Symbol of $v_2$: values on $\bigotimes$ and $\bigodot$ are needed.

Let $\Delta_n$ be the the space of all non-trivial Gauss diagrams with $n$ pts.

$\Delta_n = \text{span} \{ \bigotimes \}.$

Fix a basis invariant in $V^2$ as:

\[ v_2 (\bigotimes) = 1, \quad v_2 (\bigodot) = 0. \]

---

**Trefoils**

\[ v_2 (\text{right trefoil}) = v_2 (\text{right trefoil}) - v_2 (\text{unknot}) \]

\[ = v_2 (\text{right trefoil}) \]

\[ v_2 (\text{left trefoil}) = v_2 (\text{left trefoil}) - v_2 (\text{unknot}) \]

Thus, \[ v_2 (\text{left trefoil}) = v_2 (\text{right trefoil}) \]

Similarly, \[ v_2 (\text{right trefoil}) \parallel \]

Thus, \[ v_2 \] does not distinguish the right and left trefoils.
Similarly to order 2 invariants, we have:

\[ v_3 \left( \begin{array}{c} \circ \end{array} \right) = v_3 \left( \begin{array}{c} \bullet \end{array} \right) \]

\[ v_3 \left( \begin{array}{c} \circ \end{array} \right) = v_3 \left( \begin{array}{c} \circ \end{array} \right) \]

Thus,

\[ v_3 \left( \begin{array}{c} \circ \end{array} \right) - v_3 \left( \begin{array}{c} \circ \end{array} \right) = 0 \]

\[ v_3 \left( \begin{array}{c} \circ \end{array} \right) - v_3 \left( \begin{array}{c} \circ \end{array} \right) = v_3 \left( \begin{array}{c} \circ \end{array} \right) \]

Gauss diagram

\[ \circ \]

If \( v_3 \left( \begin{array}{c} \circ \end{array} \right) = 1 \), then the values of \( v_3 \) on the left and right trefoils differ by 1.

Thus, \( v_3 \) distinguishes the trefoils.

**Homework:**

1. Let \( v_2 \in V^2 \) be an invariant of order 2 normalized so that \( v_2 \left( \begin{array}{c} \times \end{array} \right) = 1 \), \( v_2 \left( \circ \end{array} \right) = 0 \). Compute \( v_2 \left( \begin{array}{c} \times \end{array} \right) \) (the value of this inv. of figure 8 knot).

2. Compute \( v_2 \) on the torus knot (2,5).

(You may use the diagram \( \begin{array}{c} \bigcirc \end{array} \) for this knot).

---

**Kontsevich's theorem.**

1. Vassiliev invariants of order \( n \) exist.

2. The quotient space \( V_n/ V_{n-1} \) is isomorphic to the space \( \Delta_n^* \) (i.e., the space of functions \( \Delta_n^* \) on \( \Delta_n^* \) modulo the 1-term & 4-term relations).