Problem 1. Show that there is a sequence of Reidemeister moves $R_2$ and $R_3$ transforming the diagram below into the standard diagram of the unknot.

(On each diagram, the fragment circled by ( ) is modified by the move indicated over the next arrow).
Problem 2. Compute the 3-coloring invariant of the knot 6_2.

1) Let $a, b, c$ be the colors of the three arcs meeting at the crossing 1.

2) Assume $a \neq b \neq c$. Then at crossing 2 we can determine the color of the remaining arc as $b$.

3) At crossing 3, all the arcs have color $b$.

4) Hence, same at crossing 4.

5) Crossing 5: $c = b$

6) Crossing 1 or 2: $a = c$.

Thus, $a = b = c$. \[ \Rightarrow \text{There are only 3 (trivial) 3-colorings.} \]

(Another method is to set up the $6 \times 6$ matrix describing the crossings (which arcs meet at each crossing), then finding its row-reduced echelon form and determining nullity of the matrix. The number of 3-colorings is then $3^n$ nullity.)
Problem 3. Define the bracket polynomial of a link $K$ as the polynomial in $A, B, d$ such that

$$\langle K \rangle = \Sigma_\sigma \langle K|\sigma \rangle \cdot d^{|\sigma|},$$

where $\langle K|\sigma \rangle$ is the product of labels attached to the state $\sigma$.

1. Prove the relations:
   (a) $\langle K \cup O \rangle = d \cdot \langle K \rangle$;
   (b) $\langle \tau \rangle = A \cdot \langle \chi \rangle + B \cdot \langle \tau \rangle$.

2. Using these relations, prove that

$$\langle \Omega \rangle = A \cdot B \langle \tau \rangle + (A B d + A^2 + B^2) \langle \tau \rangle$$

See lecture notes.
Problem 4. Show (by any valid method) that $4_1$ is ambient isotopic to its mirror image.

**Method 1:**
Find Reidemeister moves from $4_1$ to $4_1$, (see hw 1).

or

**Method 2:**
Compute the bracket polynomial and show (e.g., by using state sums, or the recurrence relation) that it is invariant under the change $A \leftrightarrow A^{-1}$.

or

**Method 3:**
Compute the Jones polynomial (using the skein relation) and show its invariance under $t \leftrightarrow t^{-1}$. 
Problem 5. 1. What is the Jones polynomial of the connected sum of two knots? (Find $V_{K_1 \# K_2}(t)$ in terms of $V_{K_1}(t)$ and $V_{K_2}(t)$.)

2. Let $T$ be the right trefoil. Given that $V_T(t) = t - t^3 - t^4$, compute the Jones polynomial of the following knot:

Method 1:
1. It is enough to prove that $\langle K_1 \# K_2 \rangle = \langle K_1 \rangle \cdot \langle K_2 \rangle$ (This implies that $V_{K_1 \# K_2} = V_{K_1} \cdot V_{K_2}$).

To show the statement for $\langle \rangle$, represent each of the states of $K_1 \# K_2$ in terms of the states of $K_1$ & $K_2$.

Method 2:
Use the skein relation of the Jones polynomial applied to the crossing "at the place where you take the connected sum."

Method 3:
Use induction on the number of crossings of $K_2$.

2. The given knot is the connected sum of 3 trefoils. Thus, $V_K = (t-t^3-t^4)^3$
Problem 6. Compute (inductively) the Jones polynomial of the \( n \)-component unlink.

\[ t^{-1} - t = (\sqrt{E} - \frac{1}{\sqrt{E}}) V_{L_2} \]

\[ V_{L_{\infty}} = (-\sqrt{E} - \frac{1}{\sqrt{E}}) = - (\sqrt{E} + \frac{1}{\sqrt{E}}) = - (\sqrt{E} + \frac{1}{\sqrt{E}}) V_{L_1} \]

Induction:
adding a component:

\[ V_{L_n} = - V_{L_{n-1}} \cdot (\sqrt{E} + \frac{1}{\sqrt{E}}) \]

Thus,
\[ V_{L_n} = (-1)^{n-1} (\sqrt{E} + \frac{1}{\sqrt{E}})^{n-1} \]