Lecture 6.

The Jones Polynomial

(from the bracket polynomial)

Def. The Jones polynomial $V_k(t)$ is a Laurent polynomial in $t$ assigned to an oriented link $K$ so that the following properties are satisfied:

1. $V_k(t)$ is invariant under ambient isotopy;
2. $V_k(1) = 1$;
3. $t^{-1} \cdot V_k(t^{-1}) - t \cdot V_k(t) = (tE - \frac{d}{tE}) \cdot V_k(t)$.

Q: 1) Existence? 2) Well-defined?

Define $V_k(t) = L_k (t^{-\frac{1}{4}})$, where $L_k$ is the normalized bracket polynomial.

Thm. $V_k(t)$ defined as above satisfies (1)-(3).

Proof: (1), (2) follow from the corresponding properties of $L_k(t)$. Need to check (3):

$< \langle \chi \rangle > = A \cdot < \langle \chi \rangle > + B \cdot < \chi >$.

$< \langle \chi \rangle > = B \cdot < \chi > + A \cdot < \langle \chi \rangle >$.

$B^{-1} < \langle \chi \rangle > - A^{-1} < \langle \chi \rangle > = \left( \frac{A}{B} - \frac{B}{A} \right) \cdot < \langle \chi \rangle >$.

$A \cdot < \langle \chi \rangle > - A^{-1} < \langle \chi \rangle > = \left( A^2 - A^{-2} \right) < \langle \chi \rangle >$ (compare with (3)).
Let $w = v(z)$.

Then $w(\langle \gamma \rangle) = w + 1$ and $w(\langle \gamma' \rangle) = w - 1$.

Let $\lambda = -A^3$.

Then, multiplying (*) by $\lambda^{-w}$, we get

\[ A \langle \gamma \rangle \lambda^{-w} - A^{-2} \langle \gamma' \rangle \lambda^{-w} = (A^2 - A^{-2}) \langle \gamma \rangle \lambda^{-w}. \]

\[ A \cdot \lambda \langle \gamma \rangle \lambda^{-(w+1)} - A^{-1} \cdot \lambda^{-1} \langle \gamma' \rangle \lambda^{-w} = (A^2 - A^{-2}) \langle \gamma \rangle \lambda^{-w}. \]

Putting $A = t^{-\frac{3}{4}}$, we get property (3).

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The reversing property of the Jones polynomial.

**Thm.** Let $K$ be a link, let $K_1, \ldots, K_n$ be its components.

Let $K'$ be the link obtained from $K$ by reversing the direction of one component (e.g., $K_1$).

Let $\lambda = \ell_k(K_1, K - K_1)$ be the total linking number of $K_1$ with the rest of $K$.

Then

\[ \overline{V}_{K'}(t) = t^{-3\lambda} \overline{V}_K(t). \]
The effect of reversing the direction of $K_1$ on the writhe of the link:

**Example:**

\[ ek (K_1, K_1) = \frac{1}{2} \sum_{c \in K_1 \cap K_1} \text{sign}(c) \]

\[ w(K) = w(K_1 - K_1) + w(K_1) + \Sigma \]

\[ w(K') = w(K_1 - K_1) + w(K_1) - \Sigma \]

\[ \implies w(K') = w(K) - 2 \Sigma = w(K) - 4 ek (K_1, K_1). \]

Thus,

\[ w(K') = w(K) - 4 ek (K_1, K_1) \]

\[ w(K') = w(K) - 4 \lambda. \]

Thus,

\[ L_{K'} (A) = (-A^3)^{-w(K')} \langle K' \rangle = \]

\[ = (-A^3)^{-w(K')} \langle K \rangle = \]

\[ = (-A^3)^{-w(K) + 4 \lambda} \langle K \rangle = \]

\[ = (-A^3)^{4 \lambda} L_k (A). \]

Thus,

\[ V_{K'} (t) = L_{K'} (t^{-4 \lambda}) = t^{-3 \lambda} L_k (t^{-4 \lambda}) = t^{-3 \lambda} V_k (t). \]
Thit’s conjecture: alternating diagrams are minimel.
(any knot represented by an alternating diagram cannot be represented by any other diagram with fewer crossings).
(No proof was known for about 100 years, until the discovery of the Jones polynomial).

Def. A diagram is **connected** if the underlying projection is connected subset of the plane. (Clearly, any loop diagram is connected). A diagram divides the plane into several **regions**.

**Lemma.** If the diagram is connected, all regions are homeomorphic to disc.
And \( \# \text{(regions)} = \# \text{(crossings)} + 2 \).

Def. An **isthmus** is a crossing at which there are less than 4 distinct regions:

(Isthmus is like a bridge between 2 separate diagrams. One can destroy an isthmus by flipping (the left of) the diagram: (flip the part left to the isthmus)

\[ \text{ex.} \quad \Rightarrow \quad \text{no isthmus.} \]
Def. A diagram is reduced if there are no isthmus.
(Any diagram can be made reduced by flipping a
half of diagram several times).

Def. The breadth of a Laurent polynomial is
the difference between the highest and lowest
powers of the variable.


\[ \text{Breadth} \left( t^3 + t - 17 t^{-4} \right) = 7. \]

Claim. The breadth of the bracket of the knot is an
invariant (need only consider \( R_1 \)).

Theorem.
The breadth of the bracket polynomial of a
reduced alternating diagram with \( c \) crossings is
exactly \( 4c. \)

Theorem 2.
The breadth of the bracket of any knot
diagram with \( c \) crossings is \( \leq 4c. \)

Corollary (Proof of Tait's conjecture)
Any reduced alternating diagram is minimal.

Proof of Corollary
Let \( D \) be reduced alternating diagram with \( c \) crossings.
Then breadth of the bracket of \( D = 4c. \).
But breadth is a knot invariant. Thus, by Tlm.2
there can't be any diagrams of the same
knot with fewer than \( c \) crossings.
This is equivalent to:

1. Any non-trivial reduced alternating knot diagram represents a non-trivial knot.

2. All reduced alternating diagrams of the same knot have the same # of crossings.

**Proofs of This 1 & 2:**

Recall that \( \langle k \rangle = \sum \langle k | \sigma \rangle \),

where \( \sigma \) is a state of \( k \).

If \( \sigma_A \) is a state with only A-resolution and

\[
\sigma_B \quad \text{A-resolution,} \\
\sigma_B \quad \text{B-resolution,}
\]

then

\( \sigma_A \) contributes highest power of \( A \),

\( \sigma_B \) lowest (negative) power of \( A \).

let \( |\Sigma| \) be the number of connected components in \( \sigma \) - 1 (or loops)

**Lemma 1.**

For a reduced alternating diagram \( \sigma \) with

\( \sigma_A, \sigma_B \) as above,

\[ |\sigma_A| > |\sigma_B| \]

for any state \( \sigma \), which has exactly 1 B-splitting.

**Proof**

Color (by black & white) regions at each crossing.
Split D to $G_A$. Then loops of $G_A$ are boundaries of black regions.

A-splicing with 2-shading

For $G_A$, one crossing will be different:

Since the diagram was reduced, this vertex is not an ithmus. Thus two white regions in the left diagram are different. On the right they are connected.

Thus $|G_i| = |G_A| - 1$.

Lemma 2.

D - diagram with $c$ crossings.

$G_k$ - any state with $k$ B-splicings (and $c-k$ A-splicings)

Let $G_A = G_0$, $G_1$, ..., $G_k$, $G_k$ be a chain of states (obtained from each other) and having $i$ B-splicings for $G_i$.

Then the max power in $\langle D | G_{i+1} \rangle \leq \langle D | G_i \rangle$.

Proof

$\langle D | G_i \rangle = (-1)^i A^{c-2i} (A^2 + A^{-2}) | G_i \rangle$

$\langle D | G_{i+1} \rangle = (-1)^i A^{-2i+1} (A^2 + A^{-2}) | G_{i+1} \rangle$

Power of $(A^2 + A^{-2})$ increases at most by 1. But $G_i$ dec. by 2.