Consider a link with two components, \( \alpha \) and \( \beta \). Let \( \alpha \cap \beta \) denote the set of crossings of \( \alpha \) with \( \beta \) (not including self-crossings).

Then the linking number of \( \alpha \) and \( \beta \) is

\[
\text{lk} (\alpha, \beta) = \frac{1}{2} \sum_{c \in \alpha \cap \beta} \varepsilon (c)
\]

**Thm.** \( \text{lk} \) is an invariant.

(Check how it behaves under \( R_1, R_2, R_3 \).)

**Ex. 1.** The Hopf link: two distinct possible mutual orientations

\[
\begin{align*}
\text{lk} &= -1 \\
\text{lk} &= 1
\end{align*}
\]

\[
\begin{align*}
\text{lk} &= 0
\end{align*}
\]

\( l \Rightarrow \text{The Hopf link is not equivalent to 2-component unlink.} \)

2. The White head link:

\[
\text{lk} (\alpha, \beta) = \frac{1}{2} \cdot (1+1-1-1) = 0
\]

The linking number is 0, but, clearly, the diagram suggests that linking is non-trivial!
Def. **The crossing number** - minimal number of crossings that occurs in any diagram of the knot $K$. (So far, the only crossing number we can compute is that of the unknot.)

**Operations on knots:**

**Mirror image:** for a knot $K$, its mirror image is obtained by reflecting it in a plane in $\mathbb{R}^3$. (All such reflections are equivalent!)

On a diagram, this amounts to changing all crossings to the opposite ones. (This is just a reflection in the plane of the board:)

\[ \begin{array}{c}
\includegraphics[width=0.15\textwidth]{mirror_image.png}
\end{array} \]

(We'll see that there is not eq. to its mirror image)

**Reverse:** the reverse of an oriented knot is the same knot with opposite orientation.

**Connected sum of oriented knots:** $K_1 \# K_2$ - oriented knots

Remove an arc from each, connect the ends to get a single component taking into account orientation.

\[ \begin{array}{c}
\includegraphics[width=0.2\textwidth]{connected_sum.png}
\end{array} \]

# is commutative;
\[ \bigcirc \text{ is the identity, but there is no inverse. (This later) \Rightarrow under connected sum.} \]
§3. (The linking number of a Whitehead doubling.

Example: replace knot by two parallel copies (they can twist around each other in several different ways). Add a "clasp" to join the two resulting components:

\[ \text{clasp.} \]

The linking numbers:

\[ \text{lk} = -1 \quad (\text{Full negative twisting}) \]

\[ \text{lk} = -1 \quad (\text{One copy}) \]

\[ w = -1. \]

(This is similar to two arcs forming the edge of a belt)
8x. Build a link by adding a parallel strand to a knot:

\[ \text{lk}(L) = 3. \]

If we add extra twisting, each full positive twist contributes +1.

\[ T\left( \begin{array}{c}
\text{twisting}
\end{array}\right) = 1 \]

Thus, (Whitney) The linking number of parallel twisted strands is the sum of the writhe and the twisting:

\[ \text{lk}(L) = w(K) + T(L) \]

(Note: one can now easily make up "linked" links with linking \( n = 0 \).)
3-colourings.

The number of 3-colourings of a knot is a simple computable invariant, which is defined in a combinatorial way.

Choose 3 colors to be a 3-colouring of a link. A diagram is a choice of colours for each of the arcs so that

(*) At each crossing, the three arcs that meet at the crossing are either all the same colour, or all three colours are used.

Let $D$ be a diagram and $T(D)$ be the set of 3-colourings of this diagram.

Let $\tau(D)$ be the number of 3-colourings.

$\tau(D) = 3^k$, where $k$ is the number of arcs.

Ex. 1. Standard diagram of unknot: $D = \bigcirc$

$\tau(D) = 3$.

Ex. 2. Standard diagram of the trefoil:

All three arcs meet at each of the three crossings.

$\tau(D) = 3 \cdot 3 \cdot 3 = 9$.

Ex. 3. 2-component unlink:

$\tau(D) = 3^2 = 9$.
5/4. The Hopf link (standard diagram)

\[ D = \quad \tau(D) = 3. \]

**Theorem.** The number of 3-colourings is a link invariant (i.e., is independent of the choice of a diagram representing the link).

**Proof.** Need to check the behavior under the Reidemeister moves.

**R1:**

These ends have same color

\[ \Rightarrow \]

Use this color here.

**R2:**

\[ \begin{align*}
\text{if } a + b, & \quad \text{then } c \neq a + b; \\
\text{if } a = b, & \quad \text{then } c = a = b
\end{align*} \]

**R3:**

1) all colors are the same: use the same color on the r.h.s.
2) all colors are different: use the same colours.
3) \[ g \begin{array}{c}
\text{if ends} \\
\text{are different}
\end{array} \]

**Exercise:**
Consider all cases.
Complete the proof.

**Corollary.** Trefoil is not eq. to the unknot.
8x. The Whitehead link.

2-component unlink:

Thus, everything has to be of the same color!

⇒ α distinguishes the Whitehead link from the 2-component unlink!
1. \( \lim_{x \to 0^+} \sin \frac{1}{x} \) (not an image of an injective map \( f : S^1 \to \mathbb{R}^2 \)) - not a knot (at the limit point \( * \) the derivative \( df \) is not defined).

2. \( \delta x \). Torus links:
   - \( p > 0, q \) are integers
   - form a cylinder with \( p \) strings along it:
   - twist it through \( \frac{q}{p} \) full twists:
   - connect the cylinder to become a torus:

3. Deformations of knots:
   - Knots: continuous maps \( f : S^1 \to \mathbb{R}^3 \)
   - Two knots are equivalent if the corresponding maps are homotopic:
     \( f_0 : S^1 \to \mathbb{R}^3, \ f_1 : S^1 \to \mathbb{R}^3 \)
   - are homotopic if \( \exists \) a cont. map \( F : S^1 \times I \to \mathbb{R}^3 \) s.t. \( F(t, 0) = f_0, \ F(t, 1) = f_1 \)
   - Not a good approach! All the knots are equivalent:

   But we don't want to consider these two as equivalent!
4. **Projections of knots:**

- **Regular:**

- **Irregular:**

  - or
  - by etc.

```
\[ \text{This can be resolved.} \]
```

Turns out that this irregularities can always be resolved (perturbed knot and change the direction of the proj.)