Lecture 9.

The Conway polynomial. (Refined version of the Alexander poly)

Axioms: $\nabla_k(z)$ - polynomial in $z$.

1. $\nabla_k(z)$ is an inv. of amb. iso. (For oriented links)

2. $\nabla_0 = 1$

3. $\nabla_{L^+} - \nabla_{L^-} = z \nabla_{L^0}$.

$\nabla_k(z) \in \mathbb{Z}[z] -$ the ring of polynomials in $z$ with int. coeff.

Thus, $\nabla_k(z) = a_0(k) + a_1(k)z + \ldots + \ldots$,

where each $a_n(k)$ is an inv.

From axiom 3, it follow that

$a_{n+1}(L^+\leftarrow) - a_{n+1}(L^-) = a_n(L)$. (Homework)

Assume that the axioms are consistent.

Ex. Split link (a link equivalent to the one with the diagram containing two nonempty parts that live in disjoint nhds).

E.g.: $\Rightarrow$ Whitehead $\Leftarrow$. 
Lemma. For a split link, $\nabla_\mathcal{L} = 0$.

Proof.

$L^+ \sim L^-$

$\nabla_{L^+} = \nabla_{L^-}$. Thus, $\nabla_{L^0} = 0$.

Ex.

$L^+ \sim L^-$

$\nabla_{L^0} = 0$.

More generally,

for an unlink with any number of components, $\nabla$ is 0.

This is used in recursive calculations:

$\nabla_{L^+} - \nabla_{L^-} = 2 \nabla_{L^0}$

$L^- \sim 0 \Rightarrow \nabla_{L^-} = 1 \Rightarrow \nabla_{L^+} = \nabla_{L^0} \cdot 2 + 1$.

$L^0 = L^0_+$

$L^0$

$\Theta$
\[ \nabla_{L^+} - (\nabla_{L^-} + \nabla_{L^0}) = 2 \nabla_{L^0} \]

Thus,
\[ \nabla_{L^+} = 2 \nabla_{L^0} + 1. \]

Def. Let \( c(L) = \begin{cases} 1 & \text{if } L \text{ has 1 component} \\ 0 & \text{if } L \text{ has } \geq 2 \end{cases} \)

Thm. \( a_0(K) = c(k) \) for all links \( k \) and knots.
\( a_1(K) = \begin{cases} \text{lk}(k) & \text{if } K \text{ has 2 components} \\ 0 & \text{otherwise} \end{cases} \)
Proof. Apply axiom 3.

1. For \( a_0 \):
\[
a_0(L^+) - a_0(L^-) = 0 \quad \implies \quad a_0 \text{ is inv. under strand switching.}
\]

Since \( L^+ \) can be changed into unknot (or unlink), it follows that \( a_0=\# \) or 0.
\[
\implies a_0(L) = C(L).
\]

2. For \( a_1 \):
\[
a_1(L^+) - a_1(L^-) = C(L) \quad \text{if} \quad L^+ \text{ has 2 components.}
\]

Exercise: Finish this proof.

Ex.

\( a_2 \) for the trefoil:
\[
a_2(L^+) - a_2(L^-) = \ell_k(L^o) = 1
\]
\[
\| \quad a_i(L^o)
\]
\[
a_2(L^o) = 0.
\]
\[
a_2(L^+) = 1.
\]

\( a_2(K) \) computes self-linking number

(obtained from links made by splicing crossings on \( K \).)