**Problem #27 (p. 188)**

Let \( || \cdot || \) be a norm on a real vector space \( V \) which satisfies the parallelogram law:

\[
||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2)
\]

The problem asks to prove that the following:

\[
\langle x, y \rangle = \frac{1}{4} (||x + y||^2 - ||x - y||^2)
\]

defines an inner product on \( V \), and check that with this definition, \( \langle x, x \rangle = ||x||^2 \).

To prove this, one has to verify all of the properties given in the definition of an inner product. The third and forth properties are easily proved, and I leave them to you as an exercise.

To prove the first two properties (linearity in the first argument), we need to check that

(a) \( \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \);

and

(b) \( \langle cx, y \rangle = c \langle x, y \rangle \);

To prove (a), consider the function

\[
\Phi(x, y, z) = 4 \cdot (\langle x + y, z \rangle - \langle x, z \rangle - \langle y, z \rangle)
\]

(a) is equivalent to \( \Phi(x, y, z) \equiv 0 \) for all \( x, y, z \in V \). Compute \( \Phi \) using the definition of \( \langle \cdot, \cdot \rangle \):

\[
\Phi(x, y, z) = ||x + y + z||^2 - ||x + y - z||^2 -
-||x + z||^2 + ||x - z||^2 - ||y + z||^2 + ||y - z||^2;
\]

Using the parallelogram law, we obtain:

\[
||x + y + z||^2 = 2||x + z||^2 + 2||y||^2 - ||x + z - y||^2
\]

Substituting this into (5), we obtain

\[
\Phi(x, y, z) = \frac{1}{2} (||y + z + x||^2 + ||y + z - x||^2) -
-\frac{1}{2} (||y - z + x||^2 + ||y - z - x||^2 - ||y + z||^2 + ||y - z||^2);
\]

Now let's take the expressions for \( \Phi(x, y, z) \) in (5) and (8) and compute one half of their sum. As the result, we obtain the following expression for \( \Phi(x, y, z) \):

\[
\Phi(x, y, z) = \frac{1}{2} (||y + z + x||^2 + ||y + z + x||^2) -
-\frac{1}{2} (||y - z + x||^2 + ||y - z - x||^2 - ||y + z||^2 + ||y - z||^2);
\]

By (6), the first term is equal to \( ||y + z||^2 + ||x||^2 \), and the second one is \(-||y + z||^2 - ||x||^2\). Hence, \( \Phi(x, y, z) \equiv 0 \), which proves property (a).

To prove (b), consider (for fixed \( x \) and \( y \)) the function

\[
\phi(c) = \langle cx, y \rangle - c \langle x, y \rangle
\]

By definition (2), \( \phi(0) = \frac{1}{4} (||y||^2 - ||y||^2) = 0 \) and \( \phi(-1) = 0 \), since \( \langle -x, y \rangle = -\langle x, y \rangle \). Therefore, for an integer \( n \) we get

\[
\langle nx, y \rangle = \langle (\text{sign}(n) \cdot x + \cdots + x), y \rangle = \text{sign}(n) \cdot (\langle x, y \rangle + \cdots \langle x, y \rangle) = |n| \cdot \text{sign}(n) \cdot \langle x, y \rangle = n \langle x, y \rangle
\]

Hence, \( \phi(n) = 0 \) for an integer \( n \).

Let \( p/q \) be integers, and \( q \neq 0 \). Then

\[
\langle p/q \cdot x, y \rangle = p(1/q \cdot x, y) = p/q \cdot q(1/q \cdot x, y) = p/q \langle x, y \rangle
\]
Hence, for a rational number \( c = p/q \) we have \( \varphi(c) = 0 \). It remains to prove that \( \varphi(c) \) is a continuous function, since it would then follow that it is equal to zero identically. This would imply that \( \langle cx, y \rangle = c\langle x, y \rangle \) for all \( c \) (rational or not).

You were not really expected to show continuity, but here’s the argument.

We must show that if \( c_n \to c \) is a convergent sequence, then \( \phi(c_n) \to \phi(c) \). It is clear that \( c_n\langle x, y \rangle \to c\langle x, y \rangle \), since \( \langle x, y \rangle \) is a fixed number, independent of \( n \). So we must show that \( \langle c_nx, y \rangle \to \langle cx, y \rangle \). We have

\[
\langle c_nx, y \rangle = \frac{1}{4} (\|c_nx + y\|^2 - \|c_nx - y\|^2).
\]

It is therefore enough to prove that \( \|c_nx + y\| \to \|cx + y\| \) for all \( y \) (since by replacing \( y \) with \( -y \) we would get also that \( \|c_nx - y\| \to \|cx - y\| \)).

Now, by the triangle inequality for the norm, we have

\[
\|c_nx + y\| = \|c_nx - cx + cx + y\| \leq \|c_nx - cx\| + \|cx + y\| = |c_n - c||x| + \|cx + y\|.
\]

Similarly,

\[
\|cx + y\| = \|cx - c_nx + c_nx + y\| \leq \|cx - c_nx\| + \|c_nx + y\| = |c - c_n||x| + \|c_nx + y\|.
\]

Summarizing, we get

\[
\|cx + y\| \leq |c - c_n||x| + \|c_nx + y\| \leq 2|c - c_n||x| + \|cx + y\|,
\]

or, equivalently,

\[
\|cx + y\| - |c - c_n||x| \leq \|c_nx + y\| \leq |c - c_n||x| + \|cx + y\|.
\]

Since \( c_n \to c \), we have that \( |c - c_n| \to 0 \), so that \( |c - c_n||x| \to 0 \). Thus \( \|cx + y\| - |c - c_n||x| \to 0 \) and \( |c - c_n||x| + \|cx + y\| \to \|cx + y\| \). Applying the squeeze theorem, we finally get that

\[
\|cx + y\| = \lim |c_nx + y|.
\]