Necessary and Sufficient Conditions for a Local Minimizer $x^*_k$

**Note:** We assume $f$ and $g_i$'s are twice continuously differentiable.

### Unconstrained

**Necessary**

1. $\nabla f(x) = 0$

2. $\nabla^2 f(x)$ is positive semi-definite.

**Sufficient**

1. As before

2. $\nabla^2 f(x)$ is positive definite.

### Linear Equality Constraints

**Necessary**

1. $Ax = b$

2. $Z^T \nabla f(x) = 0$ or equivalently,
   
   $\nabla f(x) = A^T \lambda$ for some $\lambda$.

3. $Z^T \nabla^2 f(x) Z$ is positive semi-definite.

**Sufficient**

1. As before

2. $Z^T \nabla^2 f(x) Z$ is positive definite.

### Linear Inequality Constraints

**Necessary**

For some Lagrange multiplier $\lambda^*_k$:

1. $Ax \geq b$

2. $\nabla f(x) = A^T \lambda$ or equivalently,
   
   $Z^T \nabla f(x) = 0$

3. $\lambda \geq 0$

4. $\lambda^T (Ax - b) = 0$

5. $Z^T \nabla^2 f(x) Z$ is positive semi-definite.

**Sufficient**

1. As before

2. $Z^T \nabla^2 f(x) Z$ is positive definite.

3. For each $i$, exactly one of $\lambda^*_i$ and $(Ax^*_i - b)_i$ is 0.

4. $Z^T \nabla^2 f(x) Z$ is positive definite.

5. $Z^T \nabla^2 f(x) Z^+$ is positive definite.

### Let $Z = \text{null-space basis matrix for } A$

### Let $A^+ = \text{submatrix of } A \text{ corresponding to the active constraints at } x^*_k$

### Let $Z^+ = \text{null-space basis matrix for } A^+$

### Let $\lambda^*_0 = \text{nonzero Lagrange multipliers}$
NOTE: WE ASSUME ANY $x_*$ IS REGULAR. THE SET
$$\{ \nabla g_i(x_*) : g_i(x_*) = 0 \}$$ IS LINEARLY INDEPENDENT

* DEFINE LAGRANGIAN:

$$L(x, \lambda) = f(x) - \lambda^T g(x) = f(x) - \sum_i \lambda_i g_i(x)$$

**NONLINEAR EQUALITY CONSTRAINTS**

**NECESSARY**

1. $g_i(x_*) = 0$

2. $\nabla_x L_i(x_*, \lambda_*) = 0$
   OR EQUIVALENTLY,
   $$Z(x_*)^T \nabla f(x_*) = 0$$

**SUFFICIENT**

1, 2 AS BEFORE

3. $Z(x_*)^T \nabla^2 x_L(x_*, \lambda_*) Z(x_*)$
   POSITIVE DEFINITE

**NONLINEAR INEQUALITY CONSTRAINTS**

**NECESSARY**

1. $g_i(x_*) \geq 0$

2. $\nabla_x L_i(x_*, \lambda_*) = 0$
   OR EQUIVALENTLY,
   $$\tilde{Z}(x_*)^T \nabla f(x_*) = 0$$

3. $\lambda_* \geq 0$

4. $\lambda_+^T g(x_*) = 0$

5. $\tilde{Z}(x_*)^T \nabla^2 x_L(x_*, \lambda_*) \tilde{Z}(x_*)$
   POS. SEMI-DEF.

**SUFFICIENT**

1, 2, 3, 4 AS BEFORE

5. $Z_+(x_*)^T \nabla^2 x_L(x_*, \lambda_*) Z_+(x_*)$
   POSITIVE DEFINITE

- LET $Z(x_*) = NULL-SPACE$ MATRIX FOR $\nabla g(x_*)^T$

- LET $\hat{g}(x) = ACTIVE$ CONSTRAINTS AT $x_*$

- LET $Z_+(x_*) = NULL-SPACE$ BASIS MATRIX FOR $\nabla g_+(x_*)^T$

- LET $g_+(x) =$ CONSTRAINTS WITH NONZERO LAGRANGE MULTIPLIERS AT $x_*$

- LET $Z_+(x_*) = NULL-SPACE$ BASIS MATRIX FOR $\nabla g_+(x_*)^T$