Midterm 2

Math 61, Section 2, Fall 2009
May 18, 2009

Read all of the following information before starting the exam:

- No calculators, cell phones, or any other electronic devices. No books or notes are allowed.
- Please leave your Student ID card out while taking the exam.
- Show all work clearly and in order. Points may be deducted if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers, when appropriate. You may leave numerical answers in un-simplified form – that is, a form which you could plug into a calculator using only numbers and “+”, “−”, “∗”, “/”, “!” and “,” and arrive at the numerical solution.
- This test has 5 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages.
- Good luck!

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<th>Problem</th>
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1. (17 points) Let $G$ be the graph with vertices $V = \{a, b, c, d, e\}$ and with adjacency matrix given by

$$
\begin{bmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0
\end{bmatrix}
$$

where the vertices are ordered alphabetically in the matrix.

   a. (1 pts) Does $G$ contain a Hamiltonian cycle? Explain your answer.

   IF A HAMILTONIAN CYCLE EXISTS, WE CAN FIND IT BY DELETING EDGES UNTIL WE GET A CONNECTED GRAPH WITH EVERY VERTEX DEGREE 2. BUT WE SEE IF WE ELIMINATE ANY EDGE INCIDENT ON C, IT CAUSES VERTEX A, B, C, OR D TO HAVE DEGREE < 2. THUS G DOES NOT CONTAIN A HAMILTONIAN CYCLE.

   b. (8 pts) Let $\hat{G}$ be the graph with vertices $\hat{V} = \{v, w, x, y, z\}$ and with adjacency matrix given by

$$
\begin{bmatrix}
0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0
\end{bmatrix}
$$

where the vertices are ordered alphabetically in the matrix. If $G$ and $\hat{G}$ are isomorphic, give a function $f : V \to \hat{V}$ that is part of an isomorphism $f, g$ from $G$ to $\hat{G}$. If not, give an invariant that is not the same between $G$ and $\hat{G}$.

\[ \text{NOT ISOMORPHIC} - \ G \ \text{CONTAINS AN EDGE OF DEGREE 4 WHEREAS} \ \hat{G} \ \text{DOES NOT.} \]
2. (25 points)

a. (10 pts) We have 30 distinct books. How many ways can the books be distributed among Persons A, B, C, and D, if Person A gets 3 books, Person B gets 15 books, Person C gets 7 books, and Person D gets 5 books? (It does not matter what order the books are handed out.)

\[
\text{NUMBER THE BOOKS } 1-30 \text{ A-D LABEL EACH ONE } A, B, C, \text{ OR } D.
\]

\[
\Rightarrow \binom{30}{3} \binom{27}{15} \binom{12}{7} \binom{5}{5} = \frac{30!}{3!15!7!5!}
\]

b. (15 pts) Suppose we now take 20 identical copies of one book and distribute them among Persons E, F, and G. How many ways can the 20 copies be distributed among Persons E, F, and G, if Person G receives less than 11 books?

**FIRST, FIND # OF WAYS TO DISTRIBUTE W/O RESTRICTIONS.**

\[
k = 20 \quad t = 3
\]

\[
(1) \quad \binom{k+t-1}{k} = \binom{22}{20} = \frac{22!}{20!2!} = \frac{22 \cdot 21}{2} = 11 \cdot 21 = 231
\]

**NEXT, FIND # OF WAYS TO DISTRIBUTE W/ G GETTING ≥ 11 BOOKS. GIVE 11 TO G THEN DISTRIBUTE THE REST W/O RESTRICTIONS:**

\[
k = 9 \quad t = 3
\]

\[
(2) \quad \binom{k+t-1}{k} = \binom{11}{9} = \frac{11!}{9!2!} = \frac{11 \cdot 10}{2} = 11 \cdot 5 = 55
\]

**THEN, # WAYS WITH G GETTING < 11 BOOKS**

\[
= (1) - (2) = 231 - 55 = 176
\]
3. (25 points) Find the general solution of the recurrence relation: \( a_n = 4a_{n-1} - 4a_{n-2} + 2^n, \quad n \geq 2. \)

**First solve** (i):

\[ a_n = 4a_{n-1} - 4a_{n-2} \]

\[ t^2 - 4t + 4 = 0 \]

\[ (t - 2)^2 = 0 \]

\[ t = 2, 2 \]

\[ \Rightarrow \quad u_n = b2^n + dn2^n \]

2) **Look for** \( g(n) \) **of same form as** **inhom.** **term:

\[ g(n) = A2^n \]

3) **Remove conflict w/** \( u_n \) **terms:

\[ g(n) = An^2 2^n \]

4) **Plug** \( g(n) \) **into** (i), **solve for** \( A_2 \):

\[ An^2 2^n = 4A(n-1)^2 2^{n-1} - 4A(n-2)^2 2^{n-2} + 2^n \]

\[ 4A n^2 2^n = 8A (n^2 - 2n + 1) 2^{n-1} - 4A (n^2 - 4n + 4) 2^{n-2} + 4 \cdot 2^n \]

\[ 0 = \left[ \frac{1}{2} A + 8A - 4A \right] n^2 2^n + \left[ \frac{-16A + 16A}{2} \right] n^2 2^n + \left[ \frac{8A - 16A + 4}{2} \right] 2^n \]

\[ \Rightarrow \quad 8A = 4 \]

\[ A = \frac{1}{2} \]

\[ \Rightarrow \quad g(n) = \frac{1}{2} n^2 2^n = n^2 2^{n-1} \]

5) \( W_n = U_n + g(n) = 162^n + dn2^n + n^2 2^{n-1} \)
4. (15 points) Let \( S_n \) denote the number of binary strings of length \( n \) that do not contain the substring "001". Find a recurrence relation defining \( S_n \) in terms of some previous value(s) \( S_{n-k} \) \((k \in \{1, 2, \ldots\})\) and a constant. You do not need to give initial conditions or solve the recurrence relation.

Every string contributing to \( S_n \) falls in one of 3 categories:

1) Starts w/ "1". Same as \( 1 \underbrace{\quad \cdots \quad}_{n-1} \)

where blank can be filled by any \( n-1 \)-length string contributing to \( S_{n-1} \).

So # of these = \( S_{n-1} \).

2) Starts w/ "01". Same as \( 01 \underbrace{\quad \cdots \quad}_{n-2} \)

where blank can be filled by any \( n-2 \)-length string contributing to \( S_{n-2} \).

So # of these = \( S_{n-2} \).

3) Starts w/ "00". The only string of this type that contributes to \( S_n \) is \( \underbrace{000 \ldots 0}_n \).

So # of these = \( 1 \).

\[ S_n = S_{n-1} + S_{n-2} + 1. \]
5. (18 points) For this problem, consider the complete bipartite graph on 500 and 2 vertices, $K_{500,2}$.

a. (10 pts) $K_{500,2}$ is planar. Find the number of faces created by $K_{500,2}$.

We can use Euler's formula for graphs:

$$f = |E| - |V| + 2$$

$|E| = 1000$  \(\Rightarrow\)  $f = 1000 - 502 + 2 = 500$

b. (8 pts) Does $K_{500,2}$ contain an Euler cycle? Justify your answer without trying to draw the graph.

For $K_{500,2}$ to contain a Euler cycle, every vertex needs to have even degree and the graph must be connected. We have 2 vertices of degree 500 and 500 vertices of degree 2, so all vertices have even degree. $K_{500,2}$ is connected by construction. Thus, it does contain an Euler cycle.