A classic problem in the theory of von Neumann algebras is the classification of II$_1$ factors $L^\infty(X) \rtimes \Gamma$ arising from free measure preserving actions $\Gamma \curvearrowright X$ of (countable discrete) groups $\Gamma$ on standard probability space $(X, \mu)$, through the so-called group measure space (or crossed product) construction of Murray and von Neumann. This problem is closely related to an area of ergodic theory which studies group actions up to orbit equivalence (OE), i.e. up to isomorphism of probability spaces $\Delta : (X, \mu) \simeq (Y, \nu)$ that takes the orbits of actions $\Gamma \curvearrowright X, \Lambda \curvearrowright Y$ onto each other, a condition translating in von Neumann algebra context into the fact that $\Delta$ extends to a $^*$-algebra isomorphism of the associated group measure space factors.

Our purpose is to give an exhaustive presentation of results in OE ergodic theory, from the origins of this subject (Murray-von Neumann 1943, I.M. Singer 1954, H. Dye 1959) until our days. The approach will be functional analytical, using von Neumann algebra framework and techniques.

We begin by producing many examples of actions (Bernoulli, Gaussian, group-like, etc) and discuss elementary properties such as freeness, ergodicity, (weak) mixing, compactness. We consider properties invariant to OE such as approximate finite dimensionality, amenability, strong ergodicity, Haagerup property, property (T), etc, and OE group-invariants: the fundamental group, the outer automorphism group, 1st and 2nd cohomology groups.

We'll then prove a celebrated theorem of Dye, Ornstein-Weiss, Connes-Feldman-Weiss (1959-1980) showing that all ergodic measure preserving actions of countable amenable groups on the non-atomic probability space are undistinguishable under orbit equivalence. Conversely, we'll show that any non-amenable $\Gamma$ has at least 2 OE inequivalent actions, with many groups having in fact uncountably many OE inequivalent actions.

Finally, we'll present several OE rigidity phenomena, showing that for special classes of (non-amenable) groups and/or actions OE is sufficient to insure isomorphism of the groups, or even conjugacy of the actions. This will include OE super-rigidity results for Bernoulli actions.

I will try to make the course as self-contained as possible, but knowledge in basic von Neumann algebras will be useful (such as the topics covered in 259A, Winter 2007). Some of the material will be presented by class participants and invited guests, during regular hours and the Monday 290I Student Seminar 4-5:30 (room MS5127). In fact, students registering for this class are encouraged to also register for the 290I class taking place Mondays 4-5:30.
The topics that I hope will be covered by students and guests during the 290I seminar are:

1. Cost of equivalence relations; Treeability and calculations of cost for free group actions (after Gaboriau).
2. $L^2$-Betti numbers for equivalence relations (after Gaboriau/Lueck, using Shlyakhtenko’s approach).
3. Construction of Gaussians and Bogoliubov actions.
5. OE rigidity results of Furman (after Furman articles).
6. OE rigidity results of Monod-Shalom (after Monod-Shalom, using Ozawa’s notes).

IMPORTANT NOTICE: Both the 259B and 290I classes will start Monday April 9’th.