An exotic sphere with positive sectional curvature

joint work with Peter Petersen

October 17, 2008
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Is the conclusion of the Sphere Theorem optimal?
Gromoll’s thesis (1966)—For each natural number $n$, there is a $\delta(n) > 0$ so that if

$$1 - \delta(n) < \sec M \leq 1,$$

then $M$ is diffeomorphic to $S^n$. 

Industry—How big is $\delta$? (Roughly every 10 years the estimate improved.)

Weiss (1993)—Not all exotic spheres admit $1/4 < \sec M$. 

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A long history of partial answers

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Much ado about nothing??

- It was possible that all of this effort was about a vacuous subject since (until now)

- there has not been a single example of an exotic sphere with positive sectional curvature.
The Gromoll-Meyer Sphere

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- A metric with this type of curvature is called *quasi-positively curved*, and

- positive curvature almost everywhere is referred to as *almost positive curvature*. 
Can a metric with *quasi*-positive curvature be deformed to one with positive curvature?
Can a metric with quasi-positive curvature be deformed to one with positive curvature?

Can a metric with almost positive curvature be deformed to one with positive curvature?
Theorem (Aubin, 1970) Any complete metric with quasi-positive Ricci curvature can be perturbed to one with positive Ricci curvature.
Tensorial Deformation

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The 2008 survey in the Bulletin of the AMS by Joachim and Wraith attributes the Ricci curvature deformation theorem to Ehrlich.
Hamilton’s Flow

- **Theorem** *(Hamilton, 1995)* Any complete metric with quasi-positive curvature operator can be perturbed to one with positive curvature operator via the Ricci flow.
**Theorem** *(Hamilton, 1995)* Any complete metric with quasi-positive curvature operator can be perturbed to one with positive curvature operator via the Ricci flow.

**Theorem** *(Böhm and Wilking 2006)* Any complete metric with positive curvature operator flows to one with constant positive curvature via the Ricci flow.
On the other hand

- So far
On the other hand

- So for

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  NOTHING
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An exotic sphere with positive curvature

Theorem The Gromoll-Meyer exotic sphere admits positive sectional curvature.
The Lie Group $Sp(2)$ can be viewed as the set of $(2 \times 2)$-matrices with quaternion entries so that

$$QQ^* = Q^* Q = id,$$

where $Q^*$ is the conjugate transpose of $Q$. 

The Gromoll–Meyer sphere, $\Sigma_7$, is the quotient of $Sp(2)$ via the free $S_3$ action $S_3 : Sp(2) \rightarrow Sp(2)$,

$$q, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} qa & qb \\ qc & qd \end{pmatrix}.$$
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- A Lie group with a biinvariant metric is nonnegatively curved.

- Combining this with O’Neill’s principle we see that the Gromoll-Meyer sphere is nonnegatively curved.
Theorem (Gromoll and Meyer, 1974) The curvature of $\Sigma^7$ at the identity is positive.

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In 1987, Strake observed that the presence of totally geodesic flat tori means that there can be no perturbation that is positive “to first order”. In particular, to prove that any deformation produces positive curvature we must consider what happens to an entire neighborhood of the old zero planes. It is not enough to just consider just what happens to the old zero planes.
Quasipositive to Almost Positive

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- In 2002, Wilking showed that all of the zero planes in $\Sigma^7$ (and most if not all) known examples with nonnegative sectional curvature are contained in totally geodesic, flat tori (also Tapp 2006).
In 1973, Cheeger discovered a way to use an isometry group $G$ to deform the metric of a nonnegatively curved manifold that tends to increase the curvature and decrease the symmetry.

- Any plane of positive curvature remains positively curved.
- Any plane whose projection to the orbits of $G$ corresponds to a positively curved plane of $G$ becomes positively curved.
- If $G = S^3$, any plane whose projection to the orbits of $G$ is nondegenerate becomes positively curved.

We go from Quasi-positive curvature to almost positive curvature on the GM-sphere via a Cheeger deformation with $S^3 \times S^4$. 
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We go from Quasi-positive curvature to almost positive curvature on the GM-sphere via a Cheeger deformation with $(S^3)^4$. 
We get almost positive curvature on the unit tangent bundle of $S^4$ by deforming the bi-invariant metric on $Sp(2)$ using Cheeger’s method and the $S^3 \times S^3 \times S^3 \times S^3$ action induced by the commuting $S^3$-actions.

\[
A_u^u \begin{pmatrix} p_1, & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} p_1 a & p_1 b \\ c & d \end{pmatrix}, \\
A_d \begin{pmatrix} p_2, & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a & b \\ p_2 c & p_2 d \end{pmatrix}, \\
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$$

We get almost positive curvature on $\Sigma^7$ as a quotient of the same metric on $Sp(2)$ (W. 2001).
The deformations that take us from the Gromoll-Meyer metric to positive curvature in the order that we perform them are:

1. The \((h_1 h_2)\)–Cheeger deformation,
2. A new deformation called the redistribution,
3. The \((U D)\)–Cheeger deformation,
4. The scaling of the \(Sp(2)\) ! \(S^4\) fibers,
5. Another new deformation called the partial conformal change,
6. The \(\Delta (U D)\) Cheeger deformation and a further \(h_1\)–deformation.
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Notation

- We let $g_1, g_{1,2}, g_{1,2,3}$, etc. be the metrics obtained after doing deformations (1), (1) and (2), or (1), (2), and (3) respectively.
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It also makes sense to talk about metrics like $g_{1,3}$, i.e. the metric obtained from doing just deformations (1) and (3) without deformation (2).
\( g_{1,3} \) is the metric that has almost positive curvature on \( \Sigma^7 \).
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• Some specific positive curvatures of $g_{1,3}$ are redistributed in $g_{1,2,3}$. The reasons for this are technical, but as far as we can tell without deformation (2) our methods will not produce positive curvature.
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• It does not seem likely that either $g_{1,2}$ or $g_{1,2,3}$ are nonnegatively curved on $Sp(2)$, but we have not verified this.
Deformation that leaves nonnegative curvature

- Deformation (4), scaling the fibers of $Sp(2) \to S^4$, does not preserve nonnegative curvature.
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- In fact this integral is positive over any of the flat tori of $g_{1,3}$.

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- In fact this integral is positive over any of the flat tori of $g_{1,3}$.

- The role of deformation (5) is to even out the positive integral.

- The curvatures of the flat tori of $g_{1,3}$ are pointwise positive with respect to $g_{1,2,3,4,5}$.
To understand the role of deformation (6), recall that we have to check that we have positive curvature not only on the 0–planes of $g_{1,3}$, but in an entire neighborhood (of uniform size) of the zero planes of $g_{1,3}$.
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To do this suppose that our zero planes have the form

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Simplifying the curvature polynomial

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We have to understand what happens when the plane is perturbed by moving its foot point, and also what happens when the plane moves within the fibers of the Grassmannian.

We’ll focus only on what happens when the plane moves within the fibers of the Grassmannian.
To do this we consider planes of the form

\[ P = \text{span} \{ \zeta + \sigma z, W + \tau V \} \]

where \( \sigma, \tau \) are real numbers and \( z \) and \( V \) are tangent vectors.

Ultimately we show that all values of all curvature polynomials \( P(\sigma, \tau) = \text{curv}(\zeta + \sigma z, W + \tau V) \) are positive.

The role of the Cheeger deformations in \( R \) is that any fixed plane with a nondegenerate projection to the vertical space of \( \Sigma \to S^4 \) becomes positively curved, provided these deformations are carried out for a sufficient long time.

Although the zero planes \( P = \text{span} \{ \zeta, W \} \) all have degenerate projections to the vertical space of \( \Sigma \to S^4 \), there are of course nearby planes whose projections are nondegenerate. Exploiting this idea we get joint work with Peter Petersen.
Moving within the Grassmannian fibers

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- The role of the Cheeger deformations in \( \delta \) is that any fixed plane with a nondegenerate projection to the vertical space of \( \Sigma^7 \rightarrow S^4 \) becomes positively curved, provided these deformations are carried out for a sufficiently long time.
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- The role of the Cheeger deformations in 6 is that any fixed plane with a nondegenerate projection to the vertical space of \(\Sigma^7 \rightarrow S^4\) becomes positively curved, provided these deformations are carried out for a sufficiently long time.

- Although the zero planes \(P = \text{span}\{\zeta, W\}\) all have degenerate projections to the vertical space of \(\Sigma^7 \rightarrow S^4\), there are of course nearby planes whose projections are nondegenerate. Exploiting this idea we get
Proposition: If all curvature polynomials whose corresponding planes have degenerate projection onto the vertical space of $\Sigma^7 \to S^4$ are positive on $\Sigma^7$, then $(\Sigma^7, g_{1,2,3,4,5})$ is positively curved, provided the Cheeger deformations in (6) are carried out for a sufficiently long time.
Proposition: If all curvature polynomials whose corresponding planes have degenerate projection onto the vertical space of $\Sigma^7 \rightarrow S^4$ are positive on $(\Sigma^7, g_{1,2,3,4,5})$, then $(\Sigma^7 g_{1,2,3,4,5,6})$ is positively curved, provided the Cheeger deformations in (6) are carried out for a sufficiently long time.

Proof: The assumptions imply that a neighborhood $N$ of the 0–locus of $g_{1,3}$ is positively curved with respect to $g_{1,2,3,4,5}$.
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Proof: The assumptions imply that a neighborhood $N$ of the 0-locus of $g_{1,3}$ is positively curved with respect to $g_{1,2,3,4,5}$.

The complement of this neighborhood is compact, so $g_{1,2,3,4,5,6}$ is positively curved on the whole complement, provided the Cheeger deformations in (6) are carried out for enough time.
**Proposition:** If all curvature polynomials whose corresponding planes have degenerate projection onto the vertical space of \( \Sigma^7 \to S^4 \) are positive on \( (\Sigma^7, g_{1,2,3,4,5}) \), then \( (\Sigma^7 g_{1,2,3,4,5,6}) \) is positively curved, provided the Cheeger deformations in (6) are carried out for a sufficiently long time.

**Proof:** The assumptions imply that a neighborhood \( N \) of the 0-locus of \( g_{1,3} \) is positively curved with respect to \( g_{1,2,3,4,5} \).

The complement of this neighborhood is compact, so \( g_{1,2,3,4,5,6} \) is positively curved on the whole complement, provided the Cheeger deformations in (6) are carried out for enough time.

Since Cheeger deformations preserve positive curvature \( g_{1,2,3,4,5,6} \) is also positively curved on \( N \). So \( g_{1,2,3,4,5,6} \) is positively curved. \( \square \)
Proposition: If all curvature polynomials whose corresponding planes have degenerate projection onto the vertical space of $\Sigma^7 \rightarrow S^4$ are positive on $(\Sigma^7, g_{1,2,3,4,5})$, then $(\Sigma^7 g_{1,2,3,4,5,6})$ is positively curved, provided the Cheeger deformations in (6) are carried out for a sufficiently long time.

Proof: The assumptions imply that a neighborhood $N$ of the 0–locus of $g_{1,3}$ is positively curved with respect to $g_{1,2,3,4,5}$.

The complement of this neighborhood is compact, so $g_{1,2,3,4,5,6}$ is positively curved on the whole complement, provided the Cheeger deformations in (6) are carried out for enough time.

Since Cheeger deformations preserve positive curvature $g_{1,2,3,4,5,6}$ is also positively curved on $N$. So $g_{1,2,3,4,5,6}$ is positively curved.

Thus the deformations in (6) allow us the computational convenience of assuming that the vector “z” is in the horizontal space of $\Sigma^7 \rightarrow S^4$.
Impossible Geometries

On the other hand, we know that the Gromoll-Meyer sphere cannot carry

- pointwise, $\frac{1}{4}$–pinched, positive curvature (Brendle and Schoen)
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- $\sec \geq 1$ and four points at pairwise distance $> \frac{\pi}{2}$ (Grove–W., 1997)
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- \( 4 + \varepsilon \geq \sec \geq 1 \) (Petersen-Tao, 2008)

joint work with Peter Petersen ()

An exotic sphere with positive sectional curva

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We still do not know whether any exotic sphere can admit

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Grove-Shiohama (1977)—such manifolds are topological spheres.
Possible Geometries

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- Grove-Shiohama (1977)–such manifolds are topological spheres.

- There are a lot more metrics with \( \sec \geq 1 \) and diameter \( > \frac{\pi}{2} \) than with
  \[ \frac{1}{4} \leq \sec < 1. \]