120A Final Instructions

November 28, 2011

The final is on Tu Dec 6 3-6pm in MS 5127 (classroom.)
Here are some previous exam problems. I’ve generally given 11-13 of these types of questions on a final. There are also similar questions on the hwk and in the book that I suggest that you look at carefully. We are covering sections 2A-F and 3A-D on this midterm. Look at problems:

Chapter 2: 1,3,4,5,6,9,10,12,13,16,17,18,19,20,21,22
Chapter 3: 2,3,5,9,11,12,13,14,15,17,18.

For the final you are allowed to bring ONE sheet of paper with your notes.
NO books or electronic devices are allowed.

I’ll have extended office hrs next week: Monday 10-Noon, 1-2pm and Tuesday 10:30-12:30.

1. Let \( \gamma(s) = f(u(s), v(s)) \) be a unit speed curve on a surface \( S \). Prove that
\[
\frac{d\nu}{ds} = -\Pi(T, T) T - \Pi(T, C) C,
\]
where \( T = \frac{d\gamma}{ds} \), \( \nu \) is the normal to \( S \), and \( C = \nu \times T \).

2. Let \( X, Y \in T_pS \) be an orthonormal basis for the tangent space at \( p \) to the surface \( S \). Prove that the mean and Gauss curvatures can be computed as follows:
\[
H = \frac{1}{2} (\Pi(X, X) + \Pi(Y, Y)),
\]
\[
K = \Pi(X, X) \Pi(Y, Y) - (\Pi(X, Y))^2.
\]

3. Let \( \alpha : (a, b) \to \mathbb{R}^3 \) be a unit speed curve with \( \kappa(s) \neq 0 \) for all \( s \in (a, b) \).
Define
\[
f(s, t) = \alpha(s) + t\alpha'(s).
\]
Prove that \( f \) defines a parametrized surface as long as \( t \neq 0 \). Compute the first and second fundamental forms and show that the Gauss curvature \( K \) vanishes.

4. For a surface of revolution \( x(t, \theta) = (r(t) \cos(\theta), r(t) \sin(\theta), z(t)) \) compute the first and second fundamental forms and the principal curvatures.
5. Let $\gamma$ be a curve on the unit sphere $S^2$. Prove that its normal curvature $\kappa_n$ is constant.

6. Let $f(u, v)$ be a parametrized surface. Recall that a tangent vector is a principal direction if it is an eigenvector for the Weingarten map. Assume that the principal curvatures are different. Show that $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ are the principal directions if and only if $F = 0 = M$.

7. Let $\alpha(u)$ be a unit speed curve in the $x, y$ plane $\mathbb{R}^2$. Show that

$$f(u, v) = (\alpha(u), v).$$

yields a parametrized surface. Compute its first and second fundamental forms and principal curvatures. Compute its Gauss curvature.

8. Show that the equation

$$ax + by + cz = d$$

defines a surface if and only if $(a, b, c) \neq (0, 0, 0)$. Show that this surface has a parametrization that is Cartesian.

9. Let $\gamma$ be a unit speed curve on a surface $S$ with normal $\nu$. Define $C = \nu \times T$, $T = \dot{\gamma}$ and

$$\kappa_g = \frac{dT}{ds} \cdot C, \quad \kappa_n = \frac{dT}{ds} \cdot \nu, \quad \tau_g = \frac{dC}{ds} \cdot \nu$$

Prove that

$$\frac{dT}{ds} = \kappa_g C + \kappa_n \nu,$$

$$\frac{dC}{ds} = -\kappa_g T + \tau_g \nu,$$

$$\frac{d\nu}{ds} = -\kappa_n T - \tau_g C.$$

10. Let $\gamma(u)$ be a regular curve in the $x, y$ plane $\mathbb{R}^2$. Show that

$$f(u, v) = (\gamma(u), v).$$

yields a parametrized surface. Compute its first fundamental form. Show that it has a Cartesian parametrization.

11. For a regular curve $\gamma(u) : I \to \mathbb{R}^3 - \{(0, 0, 0)\}$ show that $f(u, v) = v\gamma(u)$ defines a surface for $v > 0$ provided $\gamma$ and $\dot{\gamma}$ are linearly independent. Compute its first fundamental form. Show that it admits Cartesian coordinates by rewriting the surface as $f(r, \theta) = rX(\theta)$ for a suitable unit speed curve $X(\theta)$.

12. Let $f(z, \theta) = (\sqrt{1 - z^2} \cos \theta, \sqrt{1 - z^2} \sin \theta, z)$ with $-1 < z < 1$ and $-\pi < \theta < \pi$. Show that $f$ defines a parametrized surface. What is the surface?
13. Let \( f \) be a parametrized surface such that \( E = 1 \) and \( F = 0 \). Prove that the \( u \) curves are unit speed with acceleration that is perpendicular to the surface. The \( u \) curves are given by \( \gamma (u) = f (u, v) \) where \( v \) is fixed.

14. For a surface of revolution \( \sigma (t, \theta) = (r (t) \cos (\theta), r (t) \sin (\theta), z (t)) \) show that the first fundamental form is given by

\[
\begin{bmatrix}
    E & F \\
    F & G
\end{bmatrix} = \begin{bmatrix}
    \dot{r}^2 + \dot{z}^2 & 0 \\
    0 & r^2
\end{bmatrix}
\]

and that the longitudes/meridians \( \gamma (t) = \sigma ((t, \theta) \) have acceleration perpendicular to the surface provided that \( (r (t), 0, z (t)) \) is unit speed.

15. Reparametrize the curve \((r (u), z (u))\) so that the new parametrization \( \sigma (t, \theta) = (r (t) \cos (\theta), r (t) \sin (\theta), z (t)) \) is conformal.

16. Reparametrize the curve \((r (u), z (u))\) so that the new parametrization \( \sigma (t, \theta) = (r (t) \cos (\theta), r (t) \sin (\theta), z (t)) \) is equiareal.

17. Let \( f : U \to S^2 \) be a parametrization of part of the unit sphere. Show that the normal \( \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v} \) is always proportional to \( \dot{f} \).

18. Show that a Monge patch \( z = f (x, y) \) is equiareal if and only if \( f \) is constant.

19. Show that a Monge patch \( z = f (x, y) \) is conformal if and only if \( f \) is constant.

20. Show that the equation \( ax + by + cz = d \)

defines a surface if and only if \( (a, b, c) \neq (0, 0, 0) \). Show that this surface has a parametrization that is Cartesian.

21. The conoid is a special type of ruled surface given by

\[
f (t, \theta) = (r (t) \cos \theta, r (t) \sin \theta, z (\theta)) = (0, 0, z (\theta)) + r (t) (\cos \theta, \sin \theta, 0)
\]

Compute its first fundamental form. Show that if \( z (\theta) = a \theta \) for some constant \( a \), then \( r (t) \) can be reparametrized in such a way that we get a conformal parametrization.

22. Consider the two parametrized surfaces given by

\[
f_1 (\phi, u) = (\sinh \phi \cos u, \sinh \phi \sin u, u) \\
= (0, 0, u) + \sinh \phi (\cos u, \sin u, 0)
\]

\[
f_2 (t, \theta) = (\cosh t \cos \theta, \cosh t \sin \theta, t)
\]

Compute the first fundamental forms for both surfaces and show that they can be reparametrized in such a way that they have the same first
fundamental forms. (The first surface is a ruled surface with a one-to-one parametrization called the helicoid, the second surface is a surface of revolution called the catenoid.)

23. Let \( S = \{ x \in \mathbb{R}^3 : |x - m|^2 = R^2 \} \). Show that \( S \) is a surface, and that if \( I \) and \( II \) denote the first and second fundamental forms, then
\[
II = \pm \frac{1}{R} I
\]

24. The conoid is a special type of ruled surface given by
\[
f(t, \theta) = (t \cos \theta, t \sin \theta, z(\theta)) = (0, 0, z(\theta)) + t (\cos \theta, \sin \theta, 0)
\]
Compute its first and second fundamental forms as well as the Gauss and mean curvatures.

25. Let \( \gamma(t) : I \to S \) be a regular curve on a surface \( S \), with \( \nu \) being the normal to the surface. Show that
\[
\kappa_n = \frac{II(\dot{\gamma}, \dot{\gamma})}{I(\dot{\gamma}, \dot{\gamma})}, \quad \kappa_g = \frac{\det (\dot{\gamma}, \ddot{\gamma}, \nu)}{(I(\dot{\gamma}, \dot{\gamma}))^{3/2}}
\]

26. Show that the principal curvatures at a point \( p \in S \) are equal if and only if at \( p \) the mean and Gauss curvatures are related by \( H^2 = K \).

27. Compute the matrix representation of the Weingarten map for a Monge patch \( f(x, y) = (x, y, f(x, y)) \) with respect to the basis \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \).

28. Prove that if \( \alpha(s) \) is an oval (a closed planar curve with positive curvature and no self intersections), then the unit tangent field \( e_1 \) is parallel to \( e_1'' \) at four or more points.

29. Prove that the concept of a vertex for a planar curve does not depend on the parametrization.

30. Let \( c(t) \) be a closed Frenet curve in \( \mathbb{R}^3 \). Show that if its curvature is \( \leq R^{-1} \), then its length is \( \geq 2\pi R \).

31. Let \( c(t) : I \to \mathbb{R}^3 \) be a regular curve with speed \( \frac{ds}{dt} = |\frac{dc}{dt}| \), where \( s \) is the arclength parameter. Prove that
\[
\kappa = \frac{\sqrt{\frac{d^2c}{dt^2} \cdot \frac{d^2c}{dt^2} - \left( \frac{d^2s}{dt^2} \right)^2}}{\left( \frac{ds}{dt} \right)^2}
\]
32. Let \( c(t) : I \to \mathbb{R}^3 \) be a regular curve such that its unit tangent field \( e_1(t) \) is also regular. Let \( s \) be the arclength parameter for \( c \) and \( \theta \) the arclength parameter for \( e_1 \). Show that
\[
\kappa = \frac{d\theta}{ds}
\]
and
\[
\det \left( e_1, \frac{d e_1}{d \theta}, \frac{d^2 e_1}{d \theta^2} \right) = \frac{\tau}{\kappa}.
\]

33. Let \( c(t) \) be a regular curve in \( \mathbb{R}^3 \) with \( \kappa > 0 \). Prove that \( c \) is planar if and only if the triple product
\[
\begin{bmatrix}
\frac{d c}{dt} & \frac{d^2 c}{dt^2} & \frac{d^3 c}{dt^3}
\end{bmatrix} = 0
\]

34. Let \( c(s) = (x(s), y(s)) \) be a planar unit speed curve. Show that the signed curvature can be computed by
\[
\kappa = \det [c', c'']
\]

35. Let \( c(s) \) be a unit speed curve in \( \mathbb{R}^3 \) Prove that
\[
\det [c', c'', c'''] = \kappa^2 \tau.
\]
It is also possible to find formulas for
\[
\det [c'', c''', c'''']
\]
etc.

36. Let \( c(t) : I \to \mathbb{R}^3 \) be a regular curve with positive curvature. Show that the unit tangent \( e_1(t) \) is a regular and that, if \( \theta \) is an arclength parameter for \( e_1 \), then
\[
\begin{align*}
\frac{dc}{d\theta} &= \frac{1}{\kappa} e_1 \\
\frac{de_1}{d\theta} &= e_2 \\
\frac{de_2}{d\theta} &= -e_1 + \frac{\tau}{\kappa} e_3 \\
\frac{de_3}{d\theta} &= -\frac{\tau}{\kappa} e_2
\end{align*}
\]

37. Let \( c(t) : I \to \mathbb{R}^3 \) be a regular curve with positive curvature. Show that \( c \) lies in a plane if and only if the torsion vanishes.
38. Let $\gamma (\theta)$ be a simple closed planar curve with $\kappa > 0$ parametrized by $\theta$, where $\theta$ is defined as the arclength parameter of the unit tangent field $e_1$. Further assume that the width

$$w = \langle e_2 (\theta), (\gamma (\theta + \pi) - \gamma (\theta)) \rangle$$

is constant. Show that:

$$w = \frac{1}{\kappa(\theta)} + \frac{1}{\kappa(\theta + \pi)}.$$

Start by establishing the facts:

$$\frac{d\gamma}{d\theta} = \frac{1}{\kappa} e_1$$

$$\frac{de_1}{d\theta} = e_2$$

$$\frac{de_2}{d\theta} = -e_1$$

$$e_1 (\theta + \pi) = -e_1 (\theta)$$

39. Let $\alpha : (a, b) \to \mathbb{R}^3$ be a unit speed curve with $\kappa (s) \neq 0$ for all $s \in (a, b)$. Define

$$f (s, t) = \alpha (s) + t\alpha' (s).$$

Prove that $f$ defines a parametrized surface as long as $t \neq 0$. Compute the first and second fundamental forms and show that the Gauss curvature $K$ vanishes.

40. For a surface of revolution $f (t, \theta) = (r (t) \cos (\theta), r (t) \sin (\theta), z (t))$ compute the first and second fundamental forms and the principal curvatures.

41. Let $f (u, v)$ be a parametrized surface. A tangent vector is a principal direction if it is an eigenvector for the Weingarten map. Show that $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ are the principal directions if $g_{uv} = 0 = h_{uv}$, and that the principal curvatures are given by

$$h_{uu}, \quad h_{uv} / g_{uu}, \quad g_{vv}.$$

42. Let $\alpha (u)$ be a unit speed curve in the $x, y$ plane $\mathbb{R}^2$. Show that

$$\sigma (u, v) = (\alpha (u), v).$$

yields a parametrized surface. Compute its first and second fundamental forms and principal curvatures. Compute its Gauss curvature.

43. Consider a surface given by $F(x, y) = C$, i.e., it is given by a function that doesn’t depend on the third coordinate $z$. Compute the normal to this surface and show that its Gauss curvature vanishes.
44. For a regular curve \( \gamma (u) : I \to \mathbb{R}^3 - \{(0, 0, 0)\} \) show that \( f (u, v) = v \gamma (u) \) defines a surface for \( v > 0 \) provided \( \gamma \) and \( \dot{\gamma} \) are linearly independent (this a generalized cone.) Compute its first fundamental form. Show that it admits Cartesian coordinates by rewriting the surface as \( f (r, \theta) = r \delta (\theta) \) for a suitable unit speed curve \( \delta (\theta) \). Hint: \( \delta \) is the curve gotten by intersecting the generalized cone with the unit sphere.

45. Let \( f \) be a parametrization such that \( g_{uu} = 1 \) and \( g_{uv} = 0 \). Prove that the \( u \) curves are unit speed with acceleration that is perpendicular to the surface. The \( u \) curves are given by \( \gamma (u) = f (u, v) \) where \( v \) is fixed.

46. Given a surface of revolution \( \sigma_1 (r, \theta) = (r \cos \theta, r \sin \theta, z_1 (r)) \) show that there is a function \( z_2 (r) \) so that \( \sigma_1 \) becomes conformal to the cylinder \( \sigma_2 (r, \theta) = (\cos \theta, \sin \theta, z_2 (r)) \).

47. Given a surface of revolution \( \sigma_1 (r, \theta) = (r \cos \theta, r \sin \theta, z_1 (r)) \) show that there is a function \( z_2 (r) \) so that \( \sigma_1 \) becomes equiareal to the cylinder \( \sigma_2 (r, \theta) = (\cos \theta, \sin \theta, z_2 (r)) \).

48. Compute the mean curvature of the Enneper surface:

\[
\sigma_1 (r, \theta) = (r \cos \theta, r \sin \theta, z_1 (r)) \quad \text{and} \quad \sigma_2 (r, \theta) = (\cos \theta, \sin \theta, z_2 (r))
\]

Scherk minimal surface:

\[
f (u, v) = \left( u - \frac{1}{3} u^3 + uv^2, -v + \frac{1}{3} v^3 - vu^2, u^2 - v^2 \right)
\]

and Catalan surface:

\[
f (u, v) = \left( u - \sin u \cosh v, 1 - \cos u \cosh v, 4 \sin \frac{u}{2} \sinh \frac{v}{2} \right)
\]

49. Show that a generalized cylinder is minimal if and only if it is planar.

50. Show that a generalized cone is minimal if and only if it is planar.

51. Show that a surface given by \( z = f (x) + g (y) \) is a minimal if and only if

\[
\frac{d^2 f}{dx^2} + \frac{d^2 g}{dy^2} = 0
\]

52. Show that the surface given by \( \sin z = \sinh x \cosh y \) is a minimal surface.

53. Show that the surface defined by

\[
x \sin \frac{z}{a} = y \cos \frac{z}{a}
\]

is a minimal surface.
54. Let $f(u, v): U \to \mathbb{R}^3$ be a parametrized surface and $f^\epsilon (u, v) = f(u, v) + \epsilon \nu(u, v)$ the parallel surface. If $f$ is minimal, then the principal curvatures for $f^\epsilon$ satisfy that
\[
\frac{1}{\kappa_1^\epsilon} + \frac{1}{\kappa_2^\epsilon} = \text{constant}.
\]

55. Let $f(u, v): U \to \mathbb{R}^3$ be a parametrized surface such that $f$ and its Gauss map $\nu(u, v): U \to S^2$ are conformally equivalent. Show that either $f$ is minimal or parametrizing part of a sphere. Hint: Show that the surface is umbilic at any point where the mean curvature is not zero. Then use our characterization of surfaces which are totally umbilic.

56. Let $f(u, v): U \to \mathbb{R}^3$ be a parametrized surface. Show that
\[
\frac{\partial \nu}{\partial u} \times \frac{\partial \nu}{\partial v} = K \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}.
\]