120A Hwk 6

November 4, 2011

Homework problems due on Wednesday Nov. 9, 2011.

1. Let \( \alpha : (a, b) \to \mathbb{R}^3 \) be a unit speed curve with \( \kappa (s) \neq 0 \) for all \( s \in (a, b) \).

Define

\[
 f (s, t) = \alpha (s) + t\alpha' (s).
\]

Prove that \( f \) defines a parametrized surface as long as \( t \neq 0 \). Compute the first and second fundamental forms and show that the Gauss curvature \( K \) vanishes.

2. For a surface of revolution \( f (t, \theta) = (r (t) \cos (\theta), r (t) \sin (\theta), z (t)) \) compute the first and second fundamental forms and the principal curvatures.

3. Let \( f (u, v) \) be a parametrized surface. A tangent vector is a principal direction if it is an eigenvector for the Weingarten map. Show that \( \frac{\partial f}{\partial u} \) and \( \frac{\partial f}{\partial v} \) are the principal directions if \( g_{uv} = 0 = h_{uv}, \) and that the principal curvatures are given by

\[
 h_{uu}, \quad h_{vv}.
\]

4. Let \( \alpha (u) \) be a unit speed curve in the \( x, y \) plane \( \mathbb{R}^2 \). Show that

\[
 \sigma (u, v) = (\alpha (u), v).
\]

yields a parametrized surface. Compute its first and second fundamental forms and principal curvatures. Compute its Gauss curvature.

5. Consider a surface given by \( F(x, y) = C \), i.e., it is given by a function that doesn’t depend on the third coordinate \( z \). Compute the normal to this surface and show that its Gauss curvature vanishes.

6. For a regular curve \( \gamma (u) : I \to \mathbb{R}^3 - \{(0, 0, 0)\} \) show that \( f (u, v) = v\gamma (u) \) defines a surface for \( v > 0 \) provided \( \gamma \) and \( \dot{\gamma} \) are linearly independent (this a generalized cone.) Compute its first fundamental form. Show that it admits Cartesian coordinates by rewriting the surface as \( f (r, \theta) = r\delta (\theta) \) for a suitable unit speed curve \( \delta (\theta) \). Hint: \( \delta \) is the curve gotten by intersecting the generalized cone with the unit sphere.
7. Let $f$ be a parametrization such that $g_{uu} = 1$ and $g_{uv} = 0$. Prove that the $u$ curves are unit speed with acceleration that is perpendicular to the surface. The $u$ curves are given by $\gamma(u) = f(u, v)$ where $v$ is fixed.