Here is a collection of old exam problems:

1. Let $\beta: I \to \mathbb{R}^3$ be a regular curve with speed $\frac{ds}{dt} = |\frac{d\beta}{dt}|$, where $s$ is the arclength parameter. Prove that

$$\kappa = \frac{\sqrt{\frac{d^2\beta}{dt^2} \cdot \frac{d^2\beta}{dt^2} - (\frac{d^2s}{dt^2})^2}}{(\frac{ds}{dt})^2}$$

2. Let $\beta: I \to \mathbb{R}^3$ be a regular curve such that its tangent field $T(t)$ is also regular. Let $s$ be the arclength parameter for $\beta$ and $\theta$ the arclength parameter for $T$. Show that

$$\kappa = \frac{d\theta}{ds}$$

and

$$\det \begin{pmatrix} T & dT & d^2T \\ d\theta & d\theta & d\theta \\ \frac{d^2T}{d\theta^2} & \frac{d^2T}{d\theta^2} & \frac{d^2T}{d\theta^2} \end{pmatrix} = \frac{\tau}{\kappa}.$$

3. Let $\gamma(\theta)$ be an oval parametrized by $\theta$ defined by $T = (\cos \theta, \sin \theta)$ of constant width

$$w = N(\theta) \cdot (\gamma(\theta + \pi) - \gamma(\theta))$$

Show that:

$$w = \frac{1}{\kappa(\theta)} + \frac{1}{\kappa(\theta + \pi)}.$$  

You can use that

$$\frac{d\gamma}{d\theta} = \frac{1}{\kappa} T$$

$$\frac{dT}{d\theta} = N$$

$$\frac{dN}{d\theta} = -T,$$

$$N \cdot T = 0, |N| = |T| = 1$$

$$T(\theta + \pi) = -T(\theta)$$

4. Let $\alpha(s)$ unit speed curve with $\kappa > 0$. Let $\theta$ be the arclength parameter for $T = \frac{d\alpha}{ds}$. Show that the curvature satisfies:

$$\kappa = \frac{d\theta}{ds}$$

5. Prove that if $\alpha(s)$ is an oval (a closed planar curve with positive curvature and no self intersections), then the unit tangent field $T$ is parallel to $T''$ at four or more points.
6. Let $\beta(t)$ be a regular curve in $\mathbb{R}^3$ with $\kappa > 0$. Prove that $\beta$ is planar if and only if the triple product

$$\left[ \frac{d\beta}{dt}, \frac{d^2\beta}{dt^2}, \frac{d^3\beta}{dt^3} \right] \equiv 0$$

7. Let $\gamma(t) : I \rightarrow \mathbb{R}^3$ be a regular curve with positive curvature. Show that $\gamma$ lies in a plane if and only if the torsion vanishes.

8. Let $\alpha(s) = (x(s), y(s))$ be a planar unit speed curve. Show that the signed curvature can be computed by

$$\kappa = \det [\alpha', \alpha'']$$

9. Let $\alpha(s)$ be a unit speed curve in $\mathbb{R}^3$. Prove that

$$\det [\alpha', \alpha'', \alpha'''] = \kappa^2 \tau.$$  

It is also possible to find formulas for

$$\det [\alpha'', \alpha''', \alpha''''],$$

etc.

10. Prove that the concept of a vertex for a planar curve does not depend on the parametrization.

11. Let $\gamma(t) : I \rightarrow \mathbb{R}^3$ be a regular curve. Prove that

$$\kappa = \sqrt{\frac{d^2\gamma}{dt^2} \cdot \frac{d^2\gamma}{dt^2} - \left( \frac{d\gamma}{dt} \frac{d\gamma}{dt} \right)^2}$$

12. Let $\gamma(t) : I \rightarrow \mathbb{R}^3$ be a regular curve with positive curvature. Show that the unit tangent $T(t)$ is a regular and that, if $\theta$ is an arclength parameter for $T$, then

$$\frac{d\gamma}{d\theta} = \frac{1}{\kappa} T$$

$$\frac{dT}{d\theta} = N$$

$$\frac{dN}{d\theta} = -T + \frac{\tau}{\kappa} B$$

$$\frac{dB}{d\theta} = -\frac{\tau}{\kappa} B$$

13. Let $\gamma(t) : I \rightarrow \mathbb{R}^3$ be a regular curve with positive curvature. Show that $\gamma$ lies in a plane if and only if the torsion vanishes.
14. Let \( \gamma (s) = \sigma (u(s), v(s)) \) be a unit speed curve on a surface \( S \). Prove that
\[
\frac{dn}{ds} = -\Pi (T, T) T - \Pi (T, C) C,
\]
where \( T = \frac{d\gamma}{ds} \), \( n \) is the normal to \( S \), and \( C = n \times T \).

15. Let \( X, Y \in T_p S \) be an orthonormal basis for the tangent space at \( p \) to the surface \( S \). Prove that the mean and Gauss curvatures can be computed as follows:
\[
H = \frac{1}{2} \left( \Pi (X, X) + \Pi (Y, Y) \right),
\]
\[
K = \Pi (X, X) \Pi (Y, Y) - \left( \Pi (X, Y) \right)^2
\]

16. Let \( \alpha : (a, b) \to \mathbb{R}^3 \) be a unit speed curve with \( \kappa (s) \neq 0 \) for all \( s \in (a, b) \).
Define
\[
\sigma (s, t) = \alpha (s) + t\alpha' (s).
\]
Prove that \( \sigma \) defines a parametrization surface as long as \( t \neq 0 \). Compute the first and second fundamental forms and show that the Gauss curvature \( K \) vanishes.

17. For a surface of revolution \( x(t, \theta) = (r(t) \cos(\theta), r(t) \sin(\theta), z(t)) \) compute the first and second fundamental forms and the principal curvatures.

18. Let \( \gamma \) be a curve on the unit sphere \( S^2 \). Prove that its normal curvature \( \kappa_n \) is constant.

19. Let \( \sigma (u, v) \) be a parametrized surface. Recall that a tangent vector is a principal direction if it is an eigenvector for the Weingarten map. Assume that the principal curvature are different and show that \( \frac{\partial \sigma}{\partial u} \) and \( \frac{\partial \sigma}{\partial v} \) are the principal directions if and only if \( F = 0 = M \).

20. Let \( \alpha (u) \) be a unit speed curve in the \( x, y \) plane \( \mathbb{R}^2 \). Show that
\[
\sigma (u, v) = (\alpha (u), v).
\]
yields a parametrized surface. Compute its first and second fundamental forms and principal curvatures. Compute its Gauss curvature.

21. Show that the equation
\[
a x + b y + c z = d
\]
defines a surface if and only if \( (a, b, c) \neq (0, 0, 0) \). Show that this surface has a parametrization that is Cartesian.

22. Let \( \gamma \) be a unit speed curve on a surface \( S \) with normal \( N \). Define \( C = N \times T, T = \dot{\gamma} \) and
\[
\kappa_g = \frac{dT}{ds} \cdot C, \quad \kappa_n = \frac{dT}{ds} \cdot N, \quad \tau_g = \frac{dC}{ds} \cdot N
\]
Prove that
\[
\frac{dT}{ds} = \kappa_g C + \kappa_n N, \\
\frac{dC}{ds} = -\kappa_g T + \tau_g N, \\
\frac{dN}{ds} = -\kappa_n T - \tau_g C.
\]

23. Let \( \gamma (u) \) be a regular curve in the \( x, y \) plane \( \mathbb{R}^2 \). Show that \( \sigma (u, v) = (\gamma (u), v) \) yields a parametrized surface. Compute its first fundamental form and construct a local isometry from a subset of the plane to the surface.

24. For a regular curve \( \gamma (u) : I \to \mathbb{R}^3 - \{(0,0,0)\} \) show that \( \sigma (u, v) = v \gamma (u) \) defines a surface for \( v > 0 \) provided \( \gamma \) and \( \dot{\gamma} \) are linearly independent. Compute its first fundamental form. Show that it admits Cartesian coordinates by rewriting the surface as \( \sigma (r, \theta) = r \delta (\theta) \) for a suitable unit speed curve \( \delta (\theta) \).

25. Let \( \sigma (z, \theta) = (\sqrt{1 - z^2 \cos \theta}, \sqrt{1 - z^2 \sin \theta}, z) \) with \(-1 < z < 1\) and \(-\pi < \theta < \pi\). Show that \( \sigma \) defines a patch on a surface. What is the surface?

26. Let \( \sigma \) be a coordinate patch such that \( E = 1 \) and \( F = 0 \). Prove that the \( u \) curves are unit speed with acceleration that is perpendicular to the surface. The \( u \) curves are given by \( \gamma (u) = \sigma (u, v) \) where \( v \) is fixed.

27. For a surface of revolution \( \sigma (t, \theta) = (r (t) \cos \theta, r (t) \sin \theta, z (t)) \) show that the first fundamental form is given by
\[
\begin{bmatrix}
E & F \\
F & G
\end{bmatrix} = \begin{bmatrix}
\dot{r}^2 + \dot{z}^2 & 0 \\
0 & r^2
\end{bmatrix}
\]

and that the longitudes/meridians \( \gamma (t) = \sigma ((t, \theta)) \) have acceleration perpendicular to the surface provided that \( (r (t), 0, z (t)) \) is unit speed.

28. Find a conformal map from a surface of revolution \( \sigma_1 (r, \theta) = (r \cos \theta, r \sin \theta, z_1 (r)) \) to a circular cylinder \( \sigma_2 (r, \theta) = (\cos \theta, \sin \theta, z_2 (r)) \).

29. Reparametrize the curve \( (r (u), z (u)) \) so that the new parametrization \( \sigma (t, \theta) = (r (t) \cos \theta, r (t) \sin \theta, z (t)) \) is conformal.

30. Find an equiareal map from a surface of revolution \( \sigma_1 (r, \theta) = (r \cos \theta, r \sin \theta, z_1 (r)) \) to a circular cylinder \( \sigma_2 (r, \theta) = (\cos \theta, \sin \theta, z_2 (r)) \).

31. Reparametrize the curve \( (r (u), z (u)) \) so that the new parametrization \( \sigma (t, \theta) = (r (t) \cos \theta, r (t) \sin \theta, z (t)) \) is equiareal.
32. Let $\sigma : U \to S^2$ be a parametrization of part of the unit sphere. Show that the normal $\frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v}$ is always proportional to $\sigma$.

33. Show that a Monge patch $z = f(x, y)$ is equiareal if and only if $f$ is constant.

34. Show that a Monge patch $z = f(x, y)$ is conformal if and only if $f$ is constant.

35. Show that the equation $ax + by + cz = d$ defines a surface if and only if $(a, b, c) \neq (0, 0, 0)$. Show that this surface has a parametrization that is Cartesian.

36. The conoid is a special type of ruled surface given by

$$\sigma(t, \theta) = (r(t) \cos \theta, r(t) \sin \theta, z(\theta)) = (0, 0, z(\theta)) + r(t) (\cos \theta, \sin \theta, 0)$$

Compute its first fundamental form. Show that if $z(\theta) = a\theta$ for some constant $a$, then $r(t)$ can be reparametrized in such a way that we get a conformal parametrization.

37. Consider the two parametrized surfaces given by

$$\sigma_1(\phi, u) = (\sinh \phi \cos u, \sinh \phi \sin u, u)$$

$$\sigma_2(t, \theta) = (\cosh t \cos \theta, \cosh t \sin \theta, t)$$

Compute the first fundamental forms for both surfaces and construct a local isometry from the first surface to the second. (The first surface is a ruled surface with a one-to-one parametrization called the helicoid, the second surface is a surface of revolution called the catenoid.)

38. Let $S = \{ x \in \mathbb{R}^3 : |x - m|^2 = R^2 \}$. Show that $S$ is a surface, and that if $I$ and $II$ denote the first and second fundamental forms, then

$$II = \pm \frac{1}{R} I$$

39. The conoid is a special type of ruled surface given by

$$\sigma(t, \theta) = (t \cos \theta, t \sin \theta, z(\theta))$$

$$= (0, 0, z(\theta)) + t (\cos \theta, \sin \theta, 0)$$

Compute its first and second fundamental forms as well as the Gauss and mean curvatures.
40. Let $\gamma(t): I \rightarrow S$ be a regular curve on a surface $S$, with $N$ being the normal to the surface. Show that

$$\kappa_n = \frac{\Pi(\dot{\gamma}, \dot{\gamma})}{\Gamma(\dot{\gamma}, \dot{\gamma})}, \kappa_g = \frac{\det(\dot{\gamma}, \ddot{\gamma}, N)}{(\Gamma(\dot{\gamma}, \dot{\gamma}))^{3/2}}$$

41. Show that the principal curvatures at a point $p \in S$ are equal if and only if at $p$ the mean and Gauss curvatures are related by $H^2 = K$.

42. Compute the matrix representation of the Weingarten map for a Monge patch $\sigma(x, y) = (x, y, f(x, y))$ with respect to the basis $\frac{\partial \sigma}{\partial x}, \frac{\partial \sigma}{\partial y}$. 