Last time: we $A^*$, A totally ordered alphabet $A = \{1 < 2 < \cdots < n\}$

got length of longest nondecreasing subword of $w$.

Schwerdt's insertion word $w \rightarrow$ tableau $P(w)$

$$
ex \quad 13214 \quad 1 \rightarrow 13 \rightarrow 12 \rightarrow 11 \rightarrow 114
$$

Sau de taquín version

$$
ex \quad 13214 \quad 14 \rightarrow 14 \rightarrow 14 \rightarrow 14 \rightarrow 14 \rightarrow 14 \rightarrow 14 \rightarrow 14
$$

Thm 1. The maximal length of a nondecreasing subword of $w$ is the length of the first row of $P(w)$.

- The maximal length of a decreasing subword of $w$ is the length of the first column of $P(w)$.

We will prove something stronger:

let $l_k(w)$ be the maximum of the length of $k$ disjoint nondecreasing subwords of $w$.

$l_1(w) =$ maximal length of a nondecreasing subword of $w$

$l_k^r(w)$ some for decreasing subwords

$$
ex \quad 13214 \quad l_1(w) = 2, \quad l_2(w) = 4, \quad l_3(w) = 5, \quad l_4(w) = 5
$$

Thm 2 (Greene). For $w \in A^*$, if $P(w)$ has shape $\lambda = (\lambda_1 \lambda_2 \cdots \lambda_r)$

$\lambda^r = (\lambda_1^r, \lambda_2^r, \ldots, \lambda_s^r)$ then for $k = 1, \ldots, r$

$\lambda_k = \lambda_k(w) - \lambda_{k-1}(w)$

$k = 1, \ldots, s$

$\lambda_k^r = \lambda_k^r(w) - \lambda_{k-1}^r(w)$

$$
ex \quad w = 13254 \quad P(w) = 124 \quad \lambda = (3,2) \quad \lambda_1 = 3 \quad \lambda_2 = 2
$$

$\lambda^r = (3,2)$

$\lambda_3 = 5 \quad \lambda_4^r = 4 \quad \lambda_3^r = 5$

Thm 2 implies Thm 1.

To prove Thm 2 we will look at the equivalence relation in $A^*$

$u \sim v \iff P(u) = P(v)$
**Math 20k**

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ex. \( n = 2 \)

\[
\begin{align*}
12 & \rightarrow [1 \ 2] \\
21 & \rightarrow [2 \ 1] \\
11 & \rightarrow [1 \ 1]
\end{align*}
\]

so \( u \cup v \) if \( P(u) = P(v) \)

\( n = 3 \)

\[
\begin{align*}
123 & \rightarrow [1 \ 2 \ 3] \\
213 & \rightarrow [2 \ 1 \ 3] \\
132 & \rightarrow [1 \ 3 \ 2] \\
231 & \rightarrow [2 \ 3 \ 1] \\
312 & \rightarrow [3 \ 1 \ 2] \\
321 & \rightarrow [3 \ 2 \ 1]
\end{align*}
\]

\( [1 \ 2 \ 3], [2 \ 1 \ 3], [1 \ 3 \ 2], [2 \ 3 \ 1], [3 \ 1 \ 2], [3 \ 2 \ 1] \)

*That's it! The congruence in \( A^* \) is generated by relations:

\[
\begin{align*}
y x z \equiv y' x' z' & \quad (x < y' \leq z') \\
x z y \equiv x' z' y' & \quad (x < y < z')
\end{align*}
\]

\( \text{Knuth transformations} \)

This motivates the definition of the following monoid:

**Definition** The plactic monoid on \( A \) is \( Pl(A) = A^*/\equiv \) where

\[
\begin{align*}
y x z \equiv y' x' z' & \quad (x < y' \leq z') \\
x z y \equiv x' z' y' & \quad (x < y < z')
\end{align*}
\]

\( \equiv \) product \( u, v \in Pl(A) \) concatenation \( u \cdot v = uv \) modulo \( \equiv \)

**Prop 1** For \( w \in A^* \), \( \overline{w} \equiv P(w) \)

\[
\begin{align*}
\text{ex. } 132541 & \equiv P(132541) = [1 \ 4 \ 5] = \frac{14}{\ 3}
\end{align*}
\]

**Prop 2** If \( w \equiv w' \) then \( \ell_k(w) = \ell_k(w') \) for all \( k \).

**Proof of Thm 2** By Prop 1 & Prop 2 enough to show that

\[
\ell_k(t) = \lambda_1 + \lambda_2 + \cdots + \lambda_k \quad k = 1, \ldots, v \text{ for tableau } t
\]

\( \ell_k(P(w)) = \ell_k(P(w)) \)

*Take \( k \) first rows as subwords, so \( \lambda_1 + \lambda_2 + \cdots + \lambda_k \leq \ell_k(w) \)

*Conversely, a nondecreasing subword \( w \) of \( t \) uses at most one letter from each column (column is a decreasing subword). So \( k \) disjoint nondecreasing subwords use at most \( \lambda_1 + \lambda_2 + \cdots + \lambda_k \) letters.

\( \lambda_1 + \lambda_2 + \cdots + \lambda_k \leq \ell_k(w) \) so they exist.
Proof of Prop 1. Induction on |w|, true for |w| = 3
assume P(w) \equiv w need to show P(wx) \equiv wx \equiv P(w) \cdot x
enough to show the case w is a row
- if wx is a row P(wx) = wx √
- o/w P(wx) = y \cdot w'
  y (leftmost element in w) y > x
  w = \ldots y l \ldots
  then
  wx = \ldots y l \ldots x \equiv \ldots y x l \ldots (bγ (⋆))
  \equiv y \ldots x l \ldots [bγ (⋆x)]

Proof of Prop 2. Say w \& w' differ only by a length transformation
w = u x y v \\ w' = u x y v' \\ (x ≤ y < z)
* nondecreasing subwords of w' are nondecreasing subwords of w
  \ell_k(w) \geq \ell_k(w')
* let w_1, w_2, \ldots w_n nondecr. subwords of w
  w_i not nondecr. subword of w' if w_i = u' x z v'
  - if y does not occur in other w_j's, let w'_i = u' x y v'
    is nondecr. subword w'
  - if w_j = u'' y v'' then let w'_i = u'' x y v'' are nondecr.
    w_j = u'' z v'' subwords of w'
  So \ell_k(w) ≤ \ell_k(w') \Rightarrow \ell_k(w) = \ell_k(w') √

Moreover, each equivalence class in \( Pr(A) \) has exactly one tableau.
The equivalence \( N \) coincides with plactic congruence.
(i.e. w \equiv w' \ \ (Pr(w) = Pr(w')) \ \ iff \ \ w \equiv w'.)

Proof: If w \equiv w' then by Prop 1, w \equiv P(w) \equiv P(w') \equiv w'.}
If \( w \equiv w' \), by Prop 2 then \( \text{rk}(w) = \text{rk}(w') \) for all \( k \), so \( P(w) \) and \( P(w') \) have the same shape.

Let \( z \) be greatest letter in \( w \) and \( w' \) \( w = u z v \) \( w' = u' z v' \) \((w \not\equiv z \in v)\) \((w \not\equiv z \in v')\)

Claim \( u v = u' v' \)

Assume \( w \) and \( w' \) differ by one Knuth transformation
- if it does not involve \( z \) either \( u \equiv u' , v \equiv v' \) or \( u = u', v \equiv v' \)
- if it involves \( z \) deleting \( z \) in \((*)\) or \((***)\) gives \( y x = y x \) \n
so \( u v = u' v' \)

By induction on \(|w|\) we have \( P(u v) = P(u' v') \)

Now from Schensted's algorithm deleting \( z \) in \( P(u z v) \) gives \( P(u v) \)

We know where \( z \) is since we know the shape of \( P(w) \)

So \( P(w) = P(w') \)

Proof Each equivalence class in \( P(A) \) contains a unique tableau

Consequence (product revisited)

\( u, v \in P(A) \) \( u \equiv P(u) = \begin{array}{c} \begin{array}{c} p(u) \\ \end{array} \end{array} \) \( v = P(v) = \begin{array}{c} \begin{array}{c} p(v) \\ \end{array} \end{array} \)

\( u v \equiv \begin{array}{c} \begin{array}{c} p(u) \\ \end{array} \end{array} \rightarrow \begin{array}{c} \begin{array}{c} p(v) \\ \end{array} \end{array} \)

\begin{array}{c} \begin{array}{c} p(u v) \end{array} \end{array}

\[ \text{jeu-de-taquin} \]

Example \( u = 132 \ v = 54 \) \( P(u) = \begin{array}{c} \begin{array}{c} 12 \end{array} \\ \frac{3}{3} \end{array} \) \( P(v) = \begin{array}{c} \begin{array}{c} 4 \end{array} \\ \frac{5}{5} \end{array} \)

\[ 4 \rightarrow \begin{array}{c} \begin{array}{c} 12 \end{array} \\ \frac{3}{3} \end{array} \rightarrow \begin{array}{c} \begin{array}{c} 4 \end{array} \\ \frac{5}{5} \end{array} \rightarrow \begin{array}{c} \begin{array}{c} 125 \end{array} \\ \frac{35}{35} = P(132, 45) \]

\[ \begin{array}{c} \begin{array}{c} 12 \end{array} \\ \frac{3}{3} \end{array} \]