NOTES FROM 9/7

1. Homework

Homework: Section 2.2: 29, 42, 49, 53. Section 2.3: 15, 18, 30

2. The Limit Laws

Theorem 1 (The Basic Limit Laws). If \( \lim_{x \to c} f(x) \) and \( \lim_{x \to c} g(x) \) exist, and 
\[
\lim_{x \to c} f(x) = L \text{ and } \lim_{x \to c} g(x) = H,
\]
then

1. Sum Law \( \lim_{x \to c} (f(x) + g(x)) \) exists and is \( L + H \).
2. Constant Multiple Law For any number \( k \), \( \lim_{x \to c} kf(x) \) exists, and 
\[
kL = \lim_{x \to c} kf(x).
\]
3. Product Law \( \lim_{x \to c} f(x)g(x) \) exists, and 
\[
\lim_{x \to c} f(x)g(x) = LH.
\]
4. Quotient Law If \( H \neq 0 \), then 
\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{H}.
\]
5. Powers and Roots If \( n \) is a positive integer, then 
\[
\lim_{x \to c} [f(x)]^n = (\lim_{x \to c} f(x))^n,
\]
and 
\[
\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}.
\]

3. Proof from End of Class

Proof. We were going to show the first statement of Theorem 1 at the end of that class. I’ll write down the proof here. Remember, to show that the limit 
\[
\lim_{x \to c} (f(x) + g(x))
\]
exists, we must first come up with an idea for the limit, and then show that \( f(x) + g(x) \) is actually getting closer and closer to that number.

Fortunately, the statement provides a hint as to what the limit should be: we expect it is \( L + H \).

To show that is the limit, we need to show that

\[
|f(x) + g(x) - L - H|
\]
gets closer and closer to 0 as \( x \) gets closer and closer to \( c \). We use something called the triangle inequality, which says:

\[
|A + B| \leq |A| + |B|,
\]
for any numbers \( A \) and \( B \).

Exercise! Can you prove Equation (2) is true for any \( A, B \)?
Using Equation (2) let’s show that (1) is getting close to 0. Setting \( A = f(x) - L \) and \( B = g(x) - H \), we have:

\[
|f(x) + g(x) - L - H| \leq |f(x) - L| + |g(x) - H|.
\]

(3)

However, both terms on the right-hand side of equation (3) are getting arbitrary small (i.e. they are getting closer and closer to 0). That means the left hand side of equation (3) must getting arbitrarily small (since it is less than something arbitrary small).

What does that mean? Well, we have shown that \( |f(x) + g(X) - L - H| \) approaches 0 as \( x \) approaches \( c \). That means that

\[
\lim_{x \to c} (f(x) + g(x))
\]

exists, and is equal to \( L + H \). That is what we wanted to show. Then we get to write our other Latin phrase:

“Quod erat demonstrandum”

which means ”Which was what was having to be demonstrated” or something ridiculously Latin-sounding like that (this indicates the end of the proof). At the end of proofs in math we also get to put this little square to tell everyone that the proof is over.

I should say about that proof that in your calculus class you will usually -not- be expected to do proofs like that. However -it’s a good idea to be able to do such things occasionally, as it will make everything else easier, as the background ideas are stored in such proofs. Most important is just doing problems from the textbook, though.

4. Example of Use of Limit Laws

Since we didn’t get to do any examples with the Limit Laws I would like to do one here to help with the homework. Let’s do Problem 2.3.17 from the book.

Find \( \lim_{y \to 4} \frac{1}{\sqrt{6y+1}} \).

Proof. By the Quotient Law, that limit is:

\[
\lim_{y \to 4} \frac{1}{\sqrt{6y+1}} = \frac{1}{\lim_{y \to 4} \sqrt{6y+1}}.
\]

if the limit on the bottom exists. By the Power limit law, the limit on the bottom is:

\[
\sqrt{\lim_{y \to 4} 6y + 1}
\]

if the limit \( \lim_{y \to 4} 6y + 1 \) exists. However, we can see that \( \lim_{y \to 4} 6y + 1 \) exists and is equal to 25.

You should check that last sentence yourself! – There are many ways to check it, but a helpful hint is to use equation (1) from the book, which you should take great advantage of:

\[
\lim_{x \to c} x^{p/q} = c^{p/q}
\]

for integers \( p \) and \( q \) with \( q \neq 0 \).

\( \square \)