SOLUTIONS

1. Consider the basis

\[ u_1 = (2, 1, 3), \quad u_2 = (-1, 4, 5), \quad u_3 = (4, -2, 3) \]

for \( \mathbb{R}^3 \). Let \( v = (6, -1, 6) \). Which of the vectors \( u_1, u_2, u_3 \) can be replaced by \( v \) and still have a basis? Give a complete argument that your answer is correct.

**Note** \( v = u_1 + u_2 \) by inspection or by solving

\[ (6, -1, 6) = a_1(2, 1, 3) + a_2(-1, 4, 5) + a_3(4, -2, 3) \]

for \( a_1, a_2, a_3 \).

The rule in the Replacement Theorem is that if

\[ v = a_1 u_1 + a_2 u_2 + a_3 u_3, \]

we can replace \( u_2 \) by \( v \) if and only if \( a_2 = 0 \).

So as \( a_1 = 1, a_2 = 0, a_3 = 1 \), we can replace \( u_1 \) or \( u_3 \) by \( v \), but not \( u_2 \).
2. Consider the system of equations

\begin{align*}
2x_1 + 3x_2 &= b_1 \\
x_1 + 4x_2 &= b_2 \\
x_2 &= b_3 \\
x_1 - x_2 &= b_4
\end{align*}

(A) What are the relations on \((b_1, b_2, b_3, b_4)\) equivalent to the existence of a simultaneous solution \((x_1, x_2)\) of these four equations?

\textbf{SUBTRACT A MULTIPLE OF } x_2 = \frac{b_2}{3} \textbf{ FROM THE OTHER 3 EQU. TO GET}

\[2x_1 = b_1 - 3b_2, \quad x_1 = \frac{b_1 - 4b_2}{3}, \quad x_1 = \frac{b_1 - 4b_2}{3} + b_3
\]

\textbf{EXAMINATE } x_1 \textbf{ TO GET}

\[b_1 - 3b_2 = 2(b_2 - 4b_2) \text{ AND } b_1 - 4b_2 = b_2 - b_4
\]

\[x_1 = \frac{b_1 - 4b_2}{3} + b_3 = 0 \text{ AND } b_1 - 5b_2 - b_4 = 0
\]

(B) Find a basis for the space of \((b_1, b_2, b_3, b_4)\)'s for which a solution exists.

1 \text{ IF } x_1 = 1, \quad x_2 = 0, \quad (b_1, b_2, b_3, b_4) = (2, 1, 0, 1)
1 \text{ IF } x_1 = 0, \quad x_2 = 1, \quad (b_1, b_2, b_3, b_4) = (3, 2, 1, 1)

\text{Now } (x_1, x_2) = x_1 \cdot (1, 2) + x_2 \cdot (0, 1) \text{ AND This Gives } (b_1, b_2, b_3, b_4) = x_1 \cdot (2, 1, 0, 1) + x_2 \cdot (3, 2, 1, 1)

\text{So } \{(2, 1, 0, 1), (3, 2, 1, 1)\} \text{ is a BASIS}

(C) What is the dimension of the space of \((b_1, b_2, b_3, b_4)\)'s for which a solution exists? Justify your answer using part (B).

\[\text{Dim} = 2, \text{ AS WE HAVE A BASIS WITH 2 ELEMENTS.}\]
3. (A) Give a basis for the skew-symmetric 3 x 3 matrices with entries in $F$ in terms of the standard basis for $M_{3\times3}(F)$.

$$A = (a_{ij}) \quad a_{ij} = -a_{ji} \quad \forall i, j$$

$$A = a_{12}(E_{12} - E_{21}) + a_{13}(E_{13} - E_{31}) + a_{23}(E_{23} - E_{32}) \quad (a_{ii} = a_{ii} = a_{33} = 0)$$

So $E_{12} - E_{21}$, $E_{13} - E_{31}$, $E_{23} - E_{32}$ are a basis.

(B) Illustrate the Basis Extension Theorem by extending the basis you chose in part (A) to a basis for all of $M_{3\times3}(F)$.

There are many possibilities, e.g.,

$$\begin{pmatrix}
E_{12} - E_{21} & E_{13} - E_{31} & E_{13} - E_{32} & 0 & 0 & 0 & 0 & 0 & 0 \\
E_{12} - E_{21} & E_{13} - E_{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$$

It is a basis because

$$E_{12} = u_1 + u_7$$
$$E_{13} = u_2 + u_7$$
$$E_{23} = u_3 + u_7$$

So all $E_{ij} \in \text{span}(u_1, \ldots, u_9)$, so $\text{span}(u_1, \ldots, u_9) = M_{3\times3}(F)$

They are linearly independent because a maximal linearly independent subset of a spanning set is a basis, but $\text{dim}(M_{3\times3}(F)) = 9$, so we cannot have a basis with $< 9$ elements.
4. Let $V = F^4$ and let $(a_1, a_2, a_3, a_4) \in F^4$ be a non-zero vector. Let $W$ be the subspace

$$W = \{(x_1, x_2, x_3, x_4) \in V \mid a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = 0\}.$$ 

Prove that $\dim(W) = 3$ by constructing a basis for $W$. (Note: I am not asking you to prove that $W$ is a subspace, you may assume that. I am asking you to construct a basis, not to invoke the Dimension Theorem for linear transformations.)

**By relabeling the indices if necessary, assume $a_3 \neq 0$**

Now, \(-a_2/a_1, 1, 0, 0\), \(-a_3/a_1, 0, 1, 0\), \(-a_4/a_1, 0, 0, 1\) belong to $W$.

\[ \begin{align*}
&u_1 \\
u_2 \\
u_3 
\end{align*} \]

Now, \(c_1 u_1 + c_2 u_2 + c_3 u_3 = \left(\frac{-a_2 c_1 - a_3 c_2 - a_4 c_3}{a_1}, c_1, c_2, c_3\right)\)

So, \(c_1 u_1 + c_2 u_2 + c_3 u_3 = 0 \Rightarrow c_1 = 0, c_2 = 0, c_3 = 0\), so \(u_1, u_2, u_3\) are independent.

Now, \(x_1 u_1 + x_2 u_2 + x_3 u_3 = \left(\frac{-a_2 x_2 - a_3 x_3 - a_4 x_4}{a_1}, x_1, x_2, x_3, x_4\right)\)

If \((x_1, x_2, x_3, x_4)\) is any element of $W$, then \(a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = 0\). So \(x_1 = \frac{-a_2 x_2 - a_3 x_3 - a_4 x_4}{a_1}\)

So \((x_1, x_2, x_3, x_4) = x_2 u_1 + x_3 u_2 + x_4 u_3\)

So \(u_1, u_2, u_3\) span $W$, so \(u_1, u_2, u_3\) is a basis for $W$.

So \(\dim(W) = 3\), as a basis has 3 elements.