Instructions: You have 40 minutes to complete the quiz. There are two problems which are worth a total of 20 points.
You may not use any books or notes.
Write your solutions in the space below the questions. If you need more space, use the back of the page.
Do not forget to write your name and UID in the space below.

Name: ________________________________
Student ID number: ____________________

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**Problem 1. 10pts.**

Let \( a \in (0, \infty) \) and let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = \frac{1}{a + x^2} \).

Prove that there exists an \( M > 0 \) such that

\[
\forall x \in \mathbb{R}, \, \forall y \in \mathbb{R}, \, |f(x) - f(y)| \leq M|x - y|.
\]

*Help! I used the mean value theorem, and the extreme value theorem.*

*More help! At some point you should make an argument very similar to one on the quiz prep.*

*It is possible to receive more than half the points without completely solving the question. If you have a fruitful idea, express it clearly: state what you wish you could show, and prove why it would help you answer the question.*

**Solution:** Let \( M = 2 \cdot \max\{ \frac{1}{a}, \frac{1}{a^2} \} \).

Suppose \( x \in \mathbb{R} \), and note:

1. if \( |x| \leq 1 \), then \( \frac{|x|}{a(x^2)} \leq \frac{1}{a^2} \leq \frac{M}{2} \);
2. if \( |x| \geq 1 \), then \( \frac{|x|}{a(x^2)} \leq \frac{|x|}{ax^2} \leq \frac{1}{a} \leq \frac{M}{2} \).

Thus, for all \( x \in \mathbb{R} \), \( \frac{|x|}{a(x^2)} \leq \frac{M}{2} \).

Thus, for all \( x, y \in \mathbb{R} \) with \( x \neq y \), we have

\[
\left| \frac{f(x) - f(y)}{x - y} \right| = \left| \frac{\frac{1}{a + x^2} - \frac{1}{a + y^2}}{x - y} \right| = \left| \frac{x + y}{(a + x^2)(a + y^2)} \right|
\]

\[
\leq \frac{|x|}{(a + x^2)(a + y^2)} + \frac{|y|}{(a + x^2)(a + y^2)}
\]

\[
\leq \frac{|x|}{a(a + x^2)} + \frac{|y|}{a(a + y^2)} \leq M,
\]

and for all \( x, y \in \mathbb{R} \) we have \( |f(x) - f(y)| \leq M|x - y| \).

Alternatively, for \( |x| \geq 1 \), \( |f'(x)| = \frac{2|x|}{(a + x^2)^2} \leq \frac{2|x|}{x^2} \leq 2 \), and the extreme value theorem says there is an \( M' > 0 \) such that for all \( x \in [-1, 1] \), \( |f'(x)| \leq M' \).

Let \( M = \max\{2, M'\} \). Then use the mean value theorem to show this works.

I’m sorry: this question, since it didn’t strictly require either theorem I mentioned, wasn’t the best that I ever made. :(
Problem 2.

(a) [4pts.] Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = x$.

Let $P$ be the partition
$$
\left\{ 0 < \frac{1}{4} < \frac{2}{4} < \frac{3}{4} < 1 \right\}.
$$

Calculate $L(f, P)$ and $U(f, P)$.

Solution: $L(f, P) = \frac{3}{8}$, $U(f, P) = \frac{5}{8}$.

(b) [6pts.] Let $g : [0, 4] \to \mathbb{R}$ be defined by

$$
g(x) = \begin{cases} 
1 & \text{if } x \in [0, 1] \cup (3, 4]; \\
8 & \text{if } x = 2; \\
0 & \text{if } x \in (1, 2) \cup (2, 3].
\end{cases}
$$

Prove that for all $n \in \mathbb{N}$, there is a partition $P_n$ of $[0, 1]$ with fewer than 88 elements such that

$$
U(g, P_n) - L(g, P_n) \leq \frac{1}{n}.
$$

If you can’t do it with fewer than 88 elements, you’ll still get some points.

If you do it with 8 elements, you’ll get a bonus point.

Solution: Let $P_n = \{0 < 1 < 1 + \frac{1}{18n} < 2 - \frac{1}{18n} < 2 + \frac{1}{18n} < 3 < 3 + \frac{1}{18n} < 4\}$.

Then $U(g, P_n) - L(g, P_n) = \frac{1}{18n} + 8 \cdot \frac{2}{18n} + \frac{1}{18n} = \frac{1}{n}$. 