Abstract. We review and slightly generalize some definitions and results on the essential dimension.

The notion of essential dimension of an algebraic group was introduced by Buhler and Reichstein in [1] and [21]. Informally speaking, essential dimension ed(G) of an algebraic group G over a field F is the smallest number of algebraically independent parameters required to define a G-torsor over a field extension of F. Thus, the essential dimension of G measures complexity of the category of G-torsors.

More generally, the essential dimension of a functor from the category \( \text{Fields} / F \) of field extensions of \( F \) to the category \( \text{Sets} \) of sets was discussed in [2].

Let \( p \) be a prime integer. Essential \( p \)-dimension \( \text{ed}_p(G) \) of an algebraic group was introduced in [22]. The integer \( \text{ed}_p(G) \) is usually easier to calculate than \( \text{ed}(G) \), and it measures the complexity of the category of G-torsors modulo “effects of degree prime to \( p \)”.

In the present paper we study essential dimension and \( p \)-dimension of a functor \( \text{Fields} / F \to \text{Sets} \) in a uniform way (Section 1). We also introduce essential \( p \)-dimension of a class of field extensions of \( F \), or equivalently, of a detection functor \( T : \text{Fields} / F \to \text{Sets} \), i.e., a functor \( T \) with \( T(L) \) consisting of at most one element for every \( L \).

For every functor \( T : \text{Fields} / F \to \text{Sets} \), we associate the class of field extensions \( L/F \) such that \( T(L) \neq \emptyset \). The essential \( p \)-dimension of this class is called canonical \( p \)-dimension of \( T \). Note that canonical \( p \)-dimension of a detection functor was introduced in [16] with the help of so-called generic fields that are defined in terms of places of fields. We show that this notion of the canonical \( p \)-dimension coincides with ours under a mild assumption (Theorem 1.16).

In Section 2, we introduce essential \( p \)-dimension of a presheaf of sets \( S \) on the category \( \text{Var}/F \) of algebraic varieties over \( F \). We associate a functor \( \tilde{S} : \text{Fields} / F \to \text{Sets} \) to every such an \( S \), and show that \( \text{ed}_p(S) = \text{ed}_p(\tilde{S}) \) (Proposition 2.6). In practice, many functors \( \text{Fields} / F \to \text{Sets} \) are of the form \( \tilde{S} \) for some presheaf of sets \( S \). This setting allows us to define \( p \)-generic elements \( a \in S(X) \) for \( S \) and show that \( \text{ed}_p(S) = \text{ed}_p(a) \) (Theorem 2.9). Thus, to determine \( \text{ed}_p(S) \) or \( \text{ed}_p(\tilde{S}) \) it is sufficient to compute the essential \( p \)-dimension of a single generic element.

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Following the approach developed by Brosnan, Reichstein and Vistoli in [3], in Section 3 we define essential $p$-dimension of a fibered category over $\text{Var}/F$. In Section 4, we consider essential dimension of an algebraic group scheme and in Section 5 the essential $p$-dimension of finite groups. Technical results used in the paper are summarized in the Appendix.

We use the following notation:

We write $\text{Fields}/F$ for the category of finitely generated field extensions over $F$ and field homomorphisms over $F$. For any $L \in \text{Fields}/F$, we have $\text{tr. deg}_F(L) < \infty$.

In the present paper, the word “scheme” over a field $F$ means a separated scheme of finite type over $F$ and a “variety” over $F$ is an integral scheme over $F$.

Note that by definition, every variety is nonempty.

The category of algebraic varieties over $F$ is denoted by $\text{Var}/F$. For any $X \in \text{Var}/F$, the function field $F(X)$ is an object of $\text{Fields}/F$ and $\text{tr. deg} F(X) = \dim(X)$.

Let $f : X \to Y$ be a rational morphism of varieties over $F$ of the same dimension. The degree $\deg(f)$ of $f$ is zero if $f$ is not dominant and is equal to the degree of the field extension $F(X)/F(Y)$ otherwise.

An algebraic group scheme over $F$ in the paper is a group scheme of finite type over $F$.

If $R$ is a ring, we write $M(R)$ for the category of finitely generated right $R$-modules.

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1. Definition of the essential $p$-dimension

The letter $p$ in the paper denotes either a prime integer or 0. An integer $k$ is said to be prime to $p$ when $k$ is prime to $p$ if $p > 0$ and $k = 1$ if $p = 0$.

1.1. Essential $p$-dimension of a functor. Let $T : \text{Fields}/F \to \text{Sets}$ be a functor. Let $\alpha \in T(L)$ and $f : L \to L'$ a field homomorphism over $F$. The field $L'$ can be viewed as an extension of $L$ via $f$. Abusing notation we shall write $\alpha_{L'}$ for the image of $\alpha$ under the map $T(f) : T(L) \to T(L')$.

Let $K, L \in \text{Fields}/F$, $\beta \in T(K)$ and $\alpha \in T(L)$. We write $\alpha \succ_p \beta$ if there exist a finite field extension $L'$ of $L$ of degree prime to $p$ and a field homomorphism $K \to L'$ over $F$ such that $\alpha_{L'} = \beta_{L'}$. In the case $p = 0$, the relation $\alpha \succ_p \beta$ will be written as $\alpha \succ \beta$ and simply means that $L$ is an extension of $K$ with $\alpha = \beta_{L'}$.

Lemma 1.1. The relation $\succ_p$ is transitive.

Proof. Let $\alpha \in T(L)$, $\beta \in T(K)$ and $\gamma \in T(J)$. Suppose $\alpha \succ_p \beta$ and $\beta \succ_p \gamma$, i.e., there exist finite extensions $K'$ of $K$ and $L'$ of $L$, both of degree prime to $p$ and $F$-homomorphisms $J \to K'$ and $K \to L'$ such that $\alpha_{L'} = \beta_{L'}$ and $\beta_{K'} = \gamma_{K'}$. By Lemma 1.1, there is a field extension $L''/L'$ of degree prime to $p$ and a field homomorphism $K' \to L''$ extending $K \to L'$. We have $\alpha_{L''} = \beta_{L''} = \gamma_{L''}$ and $[L'' : L]$ is prime to $p$, hence $\alpha \succ_p \gamma$. \hfill $\square$

Let $K, L \in \text{Fields}/F$. An element $\alpha \in T(L)$ is said to be $p$-defined over $K$ and $K$ is a field of $p$-definition of $\alpha$ if $\alpha \succ_p \beta$ for some $\beta \in T(K)$. In the case $p = 0$, we say that $\alpha$ is defined over $K$ and $K$ is a field of definition of $\alpha$. The latter means that $L$ is an extension of $K$ and $\alpha = \beta_{L'}$ for some $\beta \in T(K)$. 

The essential $p$-dimension of $\alpha$, denoted $\text{ed}_p(\alpha)$, is the least integer $\text{tr} \cdot \deg_F(K)$ over all fields of $p$-definition $K$ of $\alpha$. In other words,
\[ \text{ed}_p(\alpha) = \min\{\text{tr} \cdot \deg_F(K)\} \]
where the minimum is taken over all fields $K/F$ such that there exists an element $\beta \in T(K)$ with $\alpha \succ_p \beta$.

The essential $p$-dimension of the functor $T$ is the integer
\[ \text{ed}_p(T) = \sup\{\text{ed}_p(\alpha)\} \]
where the supremum is taken over all $\alpha \in T(L)$ and fields $L \in \text{Fields}/F$.

We write $\text{ed}(T)$ for $\text{ed}_0(T)$ and simply call $\text{ed}(T)$ the essential dimension of $T$. Clearly, $\text{ed}(T) \geq \text{ed}_p(T)$ for all $p$.

Informally speaking, the essential dimension of $T$ is the smallest number of algebraically independent parameters required to define $T$.

An element $\alpha \in T(L)$ is called $p$-minimal if $\text{ed}_p(\alpha) = \text{tr} \cdot \deg_F(L)$, i.e., whenever $\alpha \succ_p \beta$ for some $\beta \in T(K)$, we have $\text{tr} \cdot \deg_F(K) = \text{tr} \cdot \deg_F(L)$. By Lemma [L.], for every $\alpha \in T(L)$ there is a $p$-minimal element $\beta \in T(K)$ with $\alpha \succ_p \beta$. It follows that $\text{ed}_p(T)$ is the supremum of $\text{ed}_p(\alpha)$ over all $p$-minimal elements $\alpha$.

1.2. Essential $p$-dimension of a scheme. Let $X$ be a scheme over $F$. We can view $X$ as a functor from $\text{Fields}/F$ to $\text{Sets}$ taking a field extension $L/F$ to the set of $L$-points $X(L) := \text{Mor}_F(\text{Spec} L, X)$.

**Proposition 1.2.** For any scheme $X$ over $F$, we have $\text{ed}_p(X) = \dim(X)$ for all $p$.

**Proof.** Let $\alpha : \text{Spec} L \rightarrow X$ be a point over a field $L \in \text{Fields}/F$ with image $\{x\}$. Every field of $p$-definition of $\alpha$ contains an image of the residue field $F(x)$. Moreover, $\alpha$ is $p$-defined over $F(x)$ hence $\text{ed}_p(\alpha) = \text{tr} \cdot \deg_F F(x) = \dim(x)$. It follows that $\text{ed}_p(X) = \dim(X)$. \(\square\)

1.3. Classifying variety of a functor. Let $f : S \rightarrow T$ be a morphism of functors from $\text{Fields}/F$ to $\text{Sets}$. We say that $f$ is $p$-surjective if for any field $L \in \text{Fields}/F$ and any $\alpha \in T(L)$, there is a finite field extension $L'/L$ of degree prime to $p$ such that $\alpha_{L'}$ belongs to the image of the map $S(L') \rightarrow T(L')$.

**Proposition 1.3.** Let $f : S \rightarrow T$ be a $p$-surjective morphism of functors from $\text{Fields}/F$ to $\text{Sets}$. Then $\text{ed}_p(S) \geq \text{ed}_p(T)$.

**Proof.** Let $\alpha \in T(L)$ for a field $L \in \text{Fields}/F$. By assumption, there is a finite field extension $L'/L$ of degree prime to $p$ and an element $\beta \in S(L')$ such that $f(\beta) = \alpha_{L'}$ in $T(L')$. Let $K$ be a field of $p$-definition of $\beta$, i.e., there is a field extension $L''/L'$ of degree prime to $p$, an $F$-homomorphism $K \rightarrow L''$ and an element $\gamma \in S(K)$ such that $\beta_{L''} = \gamma_{L''}$. It follows from the equality
\[ f(\gamma)_{L''} = f(\gamma_{L''}) = f(\beta_{L''}) = f(\beta)_{L''} = \alpha_{L''} \]
that $\alpha$ is $p$-defined over $K$, hence $\text{ed}_p(\beta) \geq \text{ed}_p(\alpha)$. The result follows. \(\square\)

Let $T : \text{Fields}/F \rightarrow \text{Sets}$ be a functor. A scheme $X$ over $F$ is called $p$-classifying for $T$ if there is a $p$-surjective morphism of functors $X \rightarrow T$.

Propositions [L.1] and [L.2] yield:

**Corollary 1.4.** Let $T : \text{Fields}/F \rightarrow \text{Sets}$ be a functor and let $X$ be a $p$-classifying scheme for $T$. Then $\dim(X) \geq \text{ed}_p(T)$. 
1.4. Restriction. Let \( K \in \text{Fields}/F \) and \( T : \text{Fields}/F \to \text{Sets} \) a functor. The restriction \( T_K \) of the functor \( T \) is the composition of \( T \) with the natural functor \( \text{Fields}/K \to \text{Fields}/F \) that is the identity on objects.

**Proposition 1.5.** Let \( K \in \text{Fields}/F \) and let \( T : \text{Fields}/F \to \text{Sets} \) be a functor. Then for every \( p \), we have:

1. \( \text{ed}_p(T_K) \leq \text{ed}_p(T) \).
2. If \([K : F]\) is finite and relatively prime to \( p \), then \( \text{ed}_p(T_K) = \text{ed}_p(T) \).

**Proof.** (1): Let \( \alpha \in T_K(L) \) for a field \( L \in \text{Fields}/K \). We write \( \alpha' \) for the element \( \alpha \) considered in the set \( T(L) \). Every field of \( p \)-definition of \( \alpha \) is also a field of \( p \)-definition of \( \alpha' \), hence \( \text{ed}_p(\alpha) \leq \text{ed}_p(\alpha') \) and \( \text{ed}_p(T_K) \leq \text{ed}_p(T) \).

(2): Let \( \alpha \in T(L) \) for some \( L \in \text{Fields}/F \). By Lemma 1.4, there is a field extension \( L' / L \) of degree prime to \( p \) and an \( F \)-homomorphism \( K \to L' \). As \( L' \in \text{Fields}/K \), there is a field extension \( L'' / L' \) of degree prime to \( p \), a subfield \( K' \subseteq L'' \) in \( \text{Fields}/K \) and an element \( \beta \in T(K') \) with \( \beta_{L''} = \alpha_{L''} \) and \( \text{tr}. \deg_F(K') = \text{tr}. \deg_K(K') \leq \text{ed}_p(T_K) \). Hence \( \alpha \) is \( p \)-defined over \( K' \). It follows that \( \text{ed}_p(\alpha) \leq \text{ed}_p(T_K) \) and \( \text{ed}_p(T) \leq \text{ed}_p(T_K) \).

1.5. Essential \( p \)-dimension of a class of field extensions. In this section we introduce essential \( p \)-dimension of a class of fields and relate it to the essential \( p \)-dimension of certain functors.

Let \( L \) and \( K' \) be in \( \text{Fields}/F \). We write \( L \triangleright_p K \) if there is a finite field extension \( L' / L \) of degree prime to \( p \) and a field homomorphism \( K \to L' \) over \( F \). In particular, \( L \triangleright_p K \) if \( K \subseteq L \). The relation \( \triangleright_p \) coincides with the relation introduced in Section 1.1 for the functor \( T : \text{Fields}/F \to \text{Sets} \) defined by \( T(L) = \{ L \} \) (one-element set). It follows from Lemma 1.4 that this relation is transitive.

Let \( \mathcal{C} \) be a class of fields in \( \text{Fields}/F \) closed under extensions, i.e., if \( K \in \mathcal{C} \) and \( L \in \text{Fields}/K \), then \( L \in \mathcal{C} \). For any \( L \in \mathcal{C} \), let \( \text{ed}_p^\mathcal{C}(L) \) be the least integer \( \text{tr}. \deg_F(K) \) over all fields \( K \in \mathcal{C} \) with \( L \triangleright_p K \). The **essential \( p \)-dimension of the class \( \mathcal{C} \)** is the integer

\[
\text{ed}_p(\mathcal{C}) := \sup \{ \text{ed}_p^\mathcal{C}(L) \}
\]

over all fields \( L \in \mathcal{C} \). We simply write \( \text{ed}(\mathcal{C}) \) for \( \text{ed}_p(\mathcal{C}) \) with \( p = 0 \).

Essential \( p \)-dimensions of classes of fields and functors are related as follows. Let \( \mathcal{C} \) be a class of fields in \( \text{Fields}/F \) closed under extensions. Consider the functor \( T_\mathcal{C} : \text{Fields}/F \to \text{Sets} \) defined by

\[
T_\mathcal{C}(L) = \begin{cases} 
\{ L \}, & \text{if } L \in \mathcal{C}; \\
\emptyset, & \text{otherwise.}
\end{cases}
\]

By the definition of the essential \( p \)-dimension, we have

\[
\text{ed}_p(\mathcal{C}) = \text{ed}_p(T_\mathcal{C}).
\]

Recall that a field \( L \in \mathcal{C} \), considered as an elements of \( T_\mathcal{C}(L) \), is called \( p \)-**minimal** if \( \text{ed}_p^\mathcal{C}(L) = \text{tr}. \deg_F(L) \). In other words, \( L \) is \( p \)-minimal if for any \( K \in \mathcal{C} \) with \( L \triangleright_p K \) we have \( \text{tr}. \deg_F(L) = \text{tr}. \deg_F(K) \). It follows from the definition that

\[
\text{ed}_p(\mathcal{C}) = \sup \{ \text{tr}. \deg_F(L) \}
\]

over all \( p \)-minimal fields in \( \mathcal{C} \).
The functor $T_C$ is a detection functor, i.e., a functor $T$ such that the set $T(L)$ has at most one element for every $L$. The correspondence $C \mapsto T_C$ is a bijection between classes of field extensions closed under extensions and detection functors.

1.6. Canonical $p$-dimension of a functor. Let $T : \text{Fields}/F \to \text{Sets}$ be a functor. Write $C_T$ for the class of all fields $L \in \text{Fields}/F$ such that $T(L) \neq \emptyset$. The canonical $p$-dimension $\text{cdim}_p(T)$ of the functor $T$ is the integer $\text{ed}_p(C_T)$. Equivalently, $\text{cdim}_p(T) = \text{ed}_p(T_C)$ for the detection functor $T_C$ with $C = C_T$.

In more details, for a field $L \in \text{Fields}/F$ satisfying $T(L) \neq \emptyset$ we have $\text{ed}_p^L(L)$ is the least integer $\text{tr.deg}_p K$ over all fields $K$ with $L \supseteq K$ and $T(K) \neq \emptyset$. Then

$$\text{cdim}_p(T) = \sup \{ \text{ed}_p^L(L) \}$$

over all fields $L \in \text{Fields}/F$ satisfying $T(L) \neq \emptyset$.

Note that the canonical dimension (respectively, canonical $p$-dimension) of a functor to the category of pointed sets was defined in [1] (respectively, [16]) by means of generic splitting fields. We consider a relation to generic fields in Section 1.4.

Proposition 1.6. For a functor $T : \text{Fields}/F \to \text{Sets}$, we have $\text{cdim}_p(T) \leq \text{ed}_p(T)$. If $T$ is a detection functor, then $\text{cdim}_p(T) = \text{ed}_p(T)$.

Proof. There is a (unique) natural surjective morphism $T \to T_C$ with $C = C_T$. It follows from Proposition 1.6 that $\text{cdim}_p(T) = \text{ed}_p(T_C) \leq \text{ed}_p(T)$.

Let $X$ be a scheme over $F$. Viewing $X$ as a functor from $\text{Fields}/F$ to $\text{Sets}$, we have the canonical $p$-dimension $\text{cdim}_p(X)$ of $X$ defined. In other words, $\text{cdim}_p(X)$ is the essential $p$-dimension of the class

$$C_X := \{ L \in \text{Fields}/F \text{ such that } X(L) \neq \emptyset \}.$$

By Propositions 1.6 and 1.7, $\text{cdim}_p(X) \leq \text{ed}_p(X) = \dim(X)$.

Proposition 1.7. Let $X$ be a smooth complete variety over $F$. Then $\text{cdim}_p(X)$ is the least dimension of the image of a morphism $X' \to X$, where $X'$ is a variety over $F$ admitting a dominant morphism $X' \to X$ of degree prime to $p$. In particular, $\text{cdim}(X)$ is the least dimension of the image of a rational morphism $X \dashrightarrow X$.

Proof. Let $Z \subset X$ be a closed subvariety and let $X' \to X$ and $X' \to Z$ be dominant morphisms with the first one of degree prime to $p$. Replacing $X'$ by the closure of the graph of the diagonal morphism $X' \to X \times Z$ we may assume that $X'$ is complete.

Let $L$ be in $\text{Fields}/F$ with $X(L) \neq \emptyset$ and $f : \text{Spec} L \to X$ a morphism over $F$. Let $\{x\}$ be the image of $f$. As $x$ is non-singular, there is a geometric valuation $v$ of $F(X)$ over $F$ with center $x$ and $F(v) = F(x) \subset L$ (cf. Lemma 1.4). We view $F(X)$ as a subfield of $F(X')$. As $F(X')/F(X)$ is a finite extension of degree prime to $p$, by Lemma 1.6 there is an extension $v'$ of $v$ on $F(X')$ such that $F(v')/F(v)$ is a finite extension of degree prime to $p$. Let $x'$ be the center of $v'$ on $X'$ and $z$ the image of $x'$ in $Z$. As $F(x') \subset F(v')$, the extension $F(x')/F(x)$ is finite of degree prime to $p$. Since $L \supseteq_p F(x) \supseteq_p F(z)$, we have $L \supseteq_p F(z)$ by Lemma 1.7. Therefore,

$$\text{ed}_p^L(L) \leq \text{tr.deg}_F F(z) \leq \dim(Z),$$

where $C = C_X$ and hence $\text{cdim}_p(X) \leq \dim(Z)$. 

ESSENTIAL DIMENSION 5
Conversely, note that $X$ has a point over the field $F(X)$. Choose a finite extension $L'/F(X)$ of degree prime to $p$ and a subfield $K \subset L'$ such that $X(K) \neq \emptyset$ and $\text{tr. deg}_F(K) = \text{ed}_F^p(F(X))$. Let $Z$ be the closure of the image of a point $\text{Spec } K \to X$. We have $\dim(Z) \leq \text{tr. deg}_F(K)$. The compositions $\text{Spec } L' \to \text{Spec } F(X) \to X$ and $\text{Spec } L' \to \text{Spec } K \to Z$ yield a model $X'$ of $L'$ and two dominant morphisms $X' \to X$ of degree prime to $p$ and $X' \to Z$ (cf. Appendix 6.3). We have

$$\text{cdim}_p(X) \geq \text{ed}_F^p(F(X)) = \text{tr. deg}_F(K) \geq \dim(Z).$$

As we noticed above, one has $\text{cdim}_p(X) \leq \dim(X)$ for every scheme $X$. We say that a scheme $X$ over $F$ is $p$-minimal if $\text{cdim}_p(X) = \dim(X)$. A scheme $X$ is minimal if it is $p$-minimal with $p = 0$. Every $p$-minimal scheme is minimal.

Proposition 1.1 then yields:

**Corollary 1.8.** Let $X$ be a smooth complete variety over $F$. Then

1. $X$ is $p$-minimal if and only if for any variety $X'$ over $F$ admitting a surjective morphism $X' \to X$ of degree prime to $p$, every morphism $X' \to X$ is dominant.

2. $X$ is minimal if and only if every rational morphism $X \dasharrow X$ is dominant.

Let $X$ and $Y$ be varieties over $F$ and $d = \dim(X)$. A correspondence from $X$ to $Y$, denoted $\alpha: X \twoheadrightarrow Y$, is an element $\alpha \in \text{CH}_d(X \times Y)$. If $\dim(Y) = d$, we write $\alpha^*: Y \to X$ for the image of $\alpha$ under the exchange isomorphism $\text{CH}_d(X \times Y) \simeq \text{CH}_d(Y \times X)$.

Let $\alpha: X \twoheadrightarrow Y$ be a correspondence. Assume that $Y$ is complete. The projection morphism $p: X \times Y \to X$ is proper and hence the push-forward homomorphism

$$p_*: \text{CH}_d(X \times Y) \to \text{CH}_d(X) = Z \cdot [X]$$

is defined [14, § 1.4]. The integer $\text{mult}(\alpha) \in \mathbb{Z}$ such that $p_*(\alpha) = \text{mult}(\alpha) \cdot [X]$ is called the multiplicity of $\alpha$. For example, if $\alpha$ is the class of the closure of the graph of a rational morphism $X \dasharrow Y$ of varieties of the same dimension, then $\text{mult}(\alpha) = 1$ and $\text{mult}(\alpha^i) = \text{deg}(f)$.

**Proposition 1.9.** Let $X$ be a complete variety of dimension $d$ over $F$. Suppose that for a prime integer $p$ and every correspondence $\alpha \in \text{CH}_d(X \times X)$ one has $\text{mult}(\alpha) \equiv \text{mult}(\alpha^i) \pmod{p}$. Then $X$ is $p$-minimal.

**Proof.** Let $f$ and $g : X' \to X$ be morphisms from a complete variety $X'$ of dimension $d$ and let $\alpha \in \text{CH}_d(X \times X)$ be the class of the closure of the image of $(f, g) : X' \to X \times X$. Then $\text{mult}(\alpha) = \text{deg}(f)$ and $\text{mult}(\alpha^i) = \text{deg}(g)$. Hence by assumption, $\text{deg}(f) \equiv \text{deg}(g) \pmod{p}$. If $\text{deg}(f)$ is relatively prime to $p$, then so is $\text{deg}(g)$. In particular, $g$ is dominant. By Corollary 1.1, $X$ is $p$-minimal.

**Example 1.10.** Let $q$ be a non-degenerate anisotropic quadratic form on a vector space $V$ over $F$ of dimension at least 2 and let $X$ be the associated quadratic hypersurface in $P(V)$ (cf. [13, §22]). The first Witt index $i_1(q)$ of $q$ is the Witt index of $q$ over the function field $F(X)$. It is proved in [13, Prop. 7.1] that the condition of Proposition 1.1 holds for $X$ and $p = 2$ if and only if $i_1(q) = 1$. In this case $X$ is 2-minimal. It follows that $\text{cdim}_2(X) = \text{cdim}(X) = \dim(X)$ if $i_1(q) = 1$. In general, $\text{cdim}_2(X) = \text{cdim}(X) = \dim(X) - i_1(q) + 1$ (cf. 13, Th. 7.6)).
**Example 1.11.** Let $A$ be a central simple algebra over $F$ of dimension $n^2$ and $X = SB(A)$ the Severi-Brauer variety of right ideals in $A$ of dimension $n$. In is shown in [14, Th. 2.1] that if $A$ is a division algebra of dimension a power of a prime integer $p$, then the condition of Proposition 1.10 holds for $X$ and $p$. In particular, $X$ is $p$-minimal. It follows that for any central simple algebra $A$ of $p$-primary index, we have $\text{cdim}_p(X) = \dim(X) = \text{ind}(A) - 1$. Moreover, the equality $\text{cdim}_p(X) = \text{ind}_p(A) - 1$, where $\text{ind}_p(A)$ is the largest power of $p$ dividing $\text{ind}_p(A)$, holds for every central simple algebra $A$.

This example can be generalized as follows.

**Example 1.12.** Let $p$ be a prime integer and $D$ a (finite) $p$-subgroup of the Brauer group $\text{Br}(F)$ of a field $F$. Let $A_1, A_2, \ldots, A_s$ be central simple $F$-algebras whose classes in $\text{Br}(F)$ generate $D$. Let $X = X_1 \times \cdots \times X_s$, where $X_i = SB(A_i)$ for every $i = 1, \ldots, s$. Suppose that $\dim(X)$ is the smallest possible (over all choices of the generators). Then the condition of Proposition 1.10 holds for $X$ and $p$ (cf. [14, Cor. 2.6, Rem. 2.9]) and hence $X$ is $p$-minimal.

Let $A$ be a central simple $F$-algebra of degree $n$. Consider the class $C_A$ of all splitting fields of $A$ in $\text{Fields}/F$. Let $X = SB(A)$, so $\dim(X) = n - 1$. We write $\text{cdim}_p(A)$ for $\text{cdim}_p(X)$ and $\dim(A)$ for $\text{cdim}(X)$. Since $A$ is split over a field extension $E/F$ if and only if $X(E) \neq \emptyset$, we have

$$\text{cdim}_p(A) = \dim_p(C_A) = \text{cdim}_p(X)$$

for every $p \geq 0$. Write $n = q_1q_2\cdots q_r$ where the $q_i$ are powers of distinct primes. Then $A$ is a tensor product $A_1 \otimes A_2 \otimes \cdots \otimes A_r$, where $A_i$ is a central division $F$-algebra of degree $q_i$. A field extension $E/F$ splits $A$ if and only if $E$ splits $A_i$ for all $i$. In other words, $X$ has an $E$-point if and only if the variety $Y = SB(A_1) \times SB(A_2) \times \cdots \times SB(A_r)$ has an $E$-point. Hence

$$(1) \quad \text{cdim}(A) = \text{cdim}(X) = \text{cdim}(Y) \leq \dim(Y) = \sum_{i=1}^r (q_i - 1).$$

It was conjectured in [3] that the inequality in (1) is actually an equality. This is proved in [13, Th. 2.1] (see also [14, Th. 11.4]) in the case when $r = 1$, i.e., when $\deg(A)$ is power of a prime integer. The case $n = 6$ was settled in [3].

1.7. **Canonical dimension and generic fields.** Let $F$ be a field and let $\mathcal{C}$ be a class of fields in $\text{Fields}/F$. A field $L \in \mathcal{C}$ is called $p$-generic in $\mathcal{C}$ if for any field $K \in \mathcal{C}$ there is a geometric $F$-place $L \to K'$, where $K'$ is a finite extension of $K$ of degree prime to $p$ (cf. Appendix 1.5). In the case $p = 0$ we simply say that $L$ is generic in $\mathcal{C}$. Clearly, if $L$ is generic, then it is $p$-generic for all $p$.

**Example 1.13.** If $X$ is a smooth variety, then by Lemma 1.13, the function field $F(X)$ is generic.

**Lemma 1.14.** If $L$ is a $p$-generic field in $\mathcal{C}$ and $L \to_p M$ with $M \in \mathcal{C}$, then $M$ is $p$-generic.

**Proof.** Take any $K \in \mathcal{C}$. There are field extensions $K'/K$ and $L'/L$ of degree prime to $p$, a geometric $F$-place $L \to K'$ and an $F$-homomorphism $M \to L'$. By Lemma 1.3, there is a field extension $K''/K'$ of degree prime to $p$ and a geometric
A. MERKURJEV

Let \( C \) be a variety extending the place \( L \rightarrow K' \). The composition \( M \rightarrow L' \rightarrow K'' \) is a geometric place and \( K''/K \) is an extension of degree prime to \( p \). Hence \( M \) is \( p \)-generic.

We say that a class \( C \) is closed under specializations, if for any \( F \)-place \( L \rightarrow K \) with \( L \in C \) we have \( K \in C \). Clearly if \( C \) is closed under specializations, then \( C \) is closed under extensions.

**Example 1.15.** If a variety \( X \) is complete, then the class \( C_X \) is closed under specializations. Indeed, let \( L \rightarrow K \) be an \( F \)-place with \( X(L) \neq \emptyset \). If \( R \subset L \) is the valuation ring of the place, then \( X(R) \neq \emptyset \) as \( X \) is complete. It follows that \( X(K) \neq \emptyset \) since there is an \( F \)-homomorphism \( R \rightarrow K \).

**Theorem 1.16.** Let \( C \) be a class of fields in \( \text{Fields}/F \) and \( p \geq 0 \) satisfying:

1. \( C \) has a \( p \)-generic field.
2. \( C \) is closed under specializations.

Then \( ed_p(C) \) is the least \( \text{tr.deg}_F(L) \) over all \( p \)-generic fields \( L \in C \).

**Proof.** Let \( L \in C \) be a \( p \)-generic field with the least \( \text{tr.deg}_F(L) \). By Lemma 1.14, any field \( M \in C \) with \( L \succ_p M \) is also \( p \)-generic. Hence \( L \) is \( p \)-minimal. It follows that \( \text{tr.deg}_F(L) \leq ed_p(C) \).

Let \( L \in C \) be a \( p \)-generic field and \( K \in C \) an arbitrary \( p \)-minimal field. There is a place \( L \rightarrow K' \) over \( F \), where \( K' \) is an extension of \( K \) of degree prime to \( p \). Let \( K'' \subset K' \) be the image of the place. As \( C \) is closed under specializations, we have \( K'' \in C \). Since \( K \succ_p K'' \) and \( K \) is \( p \)-minimal, we have \( \text{tr.deg}_F(K'') = \text{tr.deg}_F(K) \). Hence

\[
\text{tr.deg}_F(L) \geq \text{tr.deg}_F(K'') = \text{tr.deg}_F(K).
\]

Therefore, \( \text{tr.deg}_F(L) \geq ed_p(C) \).

**Remark 1.17.** By Examples 1.14 and 1.15, for a smooth complete variety \( X \) over \( F \), the class \( C_X \) satisfies the conditions of the theorem. In particular, for such an \( X \), the integer \( cdim_p(X) \) coincides with the canonical \( p \)-dimension introduced in 1.16.

**Example 1.18.** Let \( G \) be either a (finite) étale or a split (connected) reductive group over \( F \). Let \( B \) be a Borel subgroup in \( G \) and \( E \) a \( G \)-torsor over a field extension \( L \) of \( F \). Then \( E \) has an \( L \)-point if and only if \( E/B \) has an \( L \)-point. As \( E/B \) is a smooth complete variety, the class the class \( C_E \) satisfies the conditions of Theorem 1.16, hence \( cdim_p(E) \) can be computed using \( p \)-generic splitting fields in 1.16.

## 2. Essential \( p \)-dimension of a presheaf of sets

By a presheaf of sets on \( \text{Var}/F \) we mean a functor \( S : (\text{Var}/F)^{op} \rightarrow \text{Sets} \). If \( f : X' \rightarrow X \) is a morphism in \( \text{Var}/F \) and \( a \in S(X) \), then we often write \( a_{X'} \) for the image of \( a \) under the map \( S(f) : S(X) \rightarrow S(X') \).

**Definition 2.1.** Let \( S \) be a presheaf of sets on \( \text{Var}/F \). Let \( X, Y \in \text{Var}/F \) and \( a \in S(X), b \in S(Y) \). We write \( a \succ_p b \) if there is a variety \( X' \in \text{Var}/F \), a morphism \( g : X' \rightarrow Y \) and a dominant morphism \( f : X' \rightarrow X \) of degree prime to \( p \) such that \( a_{X'} = b_{X'} \) in \( S(X') \).
Let $S$ be a presheaf of sets on $\text{Var}/F$ and $a \in S(X)$ for some $X \in \text{Var}/F$. The essential dimension of $a$, denoted $\text{ed}_p(a)$, is the least $\dim(Y)$ over all elements $b \in S(Y)$ for a variety $Y$ with $a \succ_p b$. As $a \succ_p a$, we have $\text{ed}_p(a) \leq \dim(X)$.

The essential $p$-dimension of the functor $S$ is the integer

$$\text{ed}_p(S) = \sup \{ \text{ed}_p(a) \}$$

over all $a \in S(X)$ and varieties $X \in \text{Var}/F$. We also write $\text{ed}(S)$ for $\text{ed}_p(S)$ if $p = 0$.

The relation $\succ_p$ is not transitive in general. We refine this relation as follows. We write $a \triangleright_p b$ if $a \succ_p b$ and in addition, in Definition 2.1, the morphism $g$ is dominant. We also write $a \blacktriangleleft_p b$ if $a \succ_p b$ and in addition, in Definition 2.1, the morphism $f$ satisfies the following condition: for every point $x \in X$, there is a point $x' \in X'$ with $f(x') = x$ and $[F(x') : F(x)]$ prime to $p$.

**Lemma 2.2.** Let $S$ be a presheaf of sets on $\text{Var}/F$, $a \in S(X)$, $b \in S(Y)$ and $c \in S(Z)$.

1. If $a \succ_p b$ and $b \blacktriangleleft_p c$, then $a \succ_p c$.
2. If $a \triangleright_p b$ and $b \succ_p c$, then $a \succ_p c$.

**Proof.** In the definition of $a \succ_p b$, let $f : X' \to X$ be a dominant morphism of degree prime to $p$ and $g : X' \to Y$ a morphism. In the definition of $b \succ_p c$, let $h : Y' \to Y$ be a dominant morphism of degree prime to $p$ and $k : Y' \to Z$ a morphism. Let $y \in Y$ be the image of the generic point of $X'$ under $g$. In the case (1), there is an $y' \in Y'$ such that $f(y') = y$ and $[F(y') : F(y)]$ is prime to $p$. In the case (2), $y$ is the generic point of $Y$. If $y'$ is the generic point of $Y'$, then $[F(y') : F(y)]$ is prime to $p$. Thus in any case, $[F(y') : F(y)]$ is prime to $p$. Hence by Lemma 2.1, there is a commutative square of morphisms of varieties

$$
\begin{array}{ccc}
X' & \to & X \\
\downarrow m & & \downarrow g \\
Y' & \to & Y
\end{array}
$$

with $m$ dominant of degree prime to $p$. Then the compositions $f \circ m$ and $k \circ l$ yield $a \succ_p c$. \qed

Let $a \in S(X)$ and $V \subset X$ a subvariety. We write $a|_V$ for the restriction of $a$ on $V$.

**Lemma 2.3.** Let $S$ be a presheaf of sets on $\text{Var}/F$, $a \in S(X)$ and $b \in S(Y)$. Suppose that $a \succ_p b$. Then:

1. There is an open subvariety $U \subset X$ such that $(a|_U) \triangleright_p b$.
2. There is a closed subvariety $Z \subset Y$ such that $a \triangleright_p (b|_Z)$.

**Proof.** Choose a variety $X' \in \text{Var}/F$, a morphism $g : X' \to Y$ and a dominant morphism $f : X' \to X$ of degree prime to $p$ such that $a_{X'} = b_{X'}$ in $S(X')$.

1. By Lemma 2.2, there exists a nonempty open subset $U \subset X$ such that for every $x \in U$ there is a point $x' \in X'$ with $f(x') = x$ and the degree $[F(x') : F(x)]$ prime to $p$. Then the restrictions $f^{-1}(U) \to U$ and $f^{-1}(U) \to Y$ yield $(a|_U) \triangleright_p b$.

2. Let $Z$ be the closure of the image of $g$. We have $a \triangleright_p (b|_Z)$. \qed

**Corollary 2.4.** Let $S$ be a presheaf of sets on $\text{Var}/F$ and $a \in S(X)$. Then there is an element $b \in S(Y)$ such that $\text{ed}_p(a) = \dim(Y)$ and $a \triangleright_p b$. \qed
Proof. By the definition of the essential $p$-dimension, there is $b \in S(Y)$ such that $\text{ed}_p(a) = \dim(Y)$ and $a >_p b$. By Lemma 1, there is a closed subvariety $Z \subset Y$ such that $a \triangleright_p (b|_Z)$. In particular, $a >_p (b|_Z)$. As $\dim(Y)$ is the smallest integer with the property that $a >_p b$, we must have $\dim(Z) = \dim(Y)$, i.e., $Z = Y$. It follows that $a \triangleright_p b$. □

2.1. The associated functor $\tilde{S}$. Let $S$ be a presheaf of sets on $\text{Var}/F$. We define a functor $\tilde{S} : \text{Fields}/F \to \text{Sets}$ as follows. Let $L \in \text{Fields}/F$. The sets $S(X)$ over all models $X$ of $L$ form a direct system with respect to morphisms of models (cf. Appendix). Set

$$\tilde{S}(L) = \text{colim} S(X).$$

In particular, for any $X \in \text{Var}/F$, we have a canonical map $S(X) \to \tilde{S}(L)$ with $L = F(X)$. We write $\tilde{a}$ for an element $a \in S(X)$. For every $L \in \text{Fields}/F$, any element of $\tilde{S}(L)$ is of the form $\tilde{a}$ for some $a \in S(X)$, where $X$ is a model of $L$.

An $F$-homomorphism of fields $L \to L'$ yields a morphism $X' \to X$ of the corresponding models and hence the maps of sets $S(X) \to S(X')$ and $\tilde{S}(L) \to \tilde{S}(L')$ making $\tilde{S}$ a functor.

Recall that we have the relations $>_p$ and $\triangleright_p$ defined for the functors $S$ and $\tilde{S}$ respectively.

Lemma 2.5. Let $S$ be a presheaf of sets on $\text{Var}/F$, $X \in \text{Var}/F$, $K \in \text{Fields}/F$ and $a \in S(X)$ and $\beta \in \tilde{S}(K)$. Then $\tilde{a} >_p \beta$ if and only if there is a model $Y$ of $K$ and an element $b \in S(Y)$ such that $\tilde{b} = \beta$ and $a \triangleright_p b$.

Proof. $\Rightarrow$: There is a finite field extension $L'/F(X)$ of degree prime to $p$ and an $F$-homomorphism $K \to L'$ such that $\tilde{a}_{L'} = \beta_{L'}$. One can choose a model $X'$ of $L'$ and $Y$ of $K$ together with two dominant morphisms $X' \to X$ and $X' \to Y$, the first of degree prime to $p$, that induce field homomorphisms $F(X) \to L'$ and $K \to L'$ respectively. Replacing $Y$ and $X'$ by open subvarieties, we may assume that there is $b \in S(Y)$ with $\tilde{b} = \beta$. The elements $a_{X'}$ and $b_{X'}$ may not be equal in $S(X')$ but they coincide when restricted to an open subvariety $U \subset X'$. Replacing $X'$ by $U$, the variety $Y$ by an open subvariety $W$ in the image of $U$ and $b$ by $b|_W$ we get the $a \triangleright_p b$.

$\Leftarrow$: Choose a variety $X' \in \text{Var}/F$, a dominant morphism $g : X' \to Y$ and a dominant morphism $f : X' \to X$ of degree prime to $p$ such that $a_{X'} = b_{X'}$ in $S(X')$. Then $F(Y)$ and $F(X')$ are subfields of $F(X')$, the degree $[F(X') : F(X)]$ is prime to $p$ and $\tilde{a}_{F(X')} = \tilde{b}_{F(X')} = \beta_{F(X')}$, hence $\tilde{a} >_p \beta$. □

Proposition 2.6. Let $S$ be a presheaf of sets on $\text{Var}/F$, $X \in \text{Var}/F$ and $a \in S(X)$. Then $\text{ed}_p(a) = \text{ed}_p(\tilde{a})$ for all $p$. Moreover, $\text{ed}_p(S) = \text{ed}_p(\tilde{S})$.

Proof. By Corollary, there is $b \in S(Y)$ such that $\text{ed}_p(a) = \dim(Y)$ and $a \triangleright_p b$. It follows from Lemma that $\tilde{a} >_p \tilde{b}$. Hence

$$\text{ed}_p(\tilde{a}) \leq \text{tr. deg}_F F(Y) = \dim(Y) = \text{ed}_p(a).$$

Let $\beta \in \tilde{S}(L)$ be so that $\tilde{a} >_p \beta$ and $\text{ed}_p(\tilde{a}) = \text{tr. deg}_F(L)$. By Lemma we can choose a model $Y$ of $L$ and an element $b \in S(Y)$ so that $\tilde{b} = \beta$ and $a \triangleright_p b$. Hence

$$\text{ed}_p(a) \leq \dim(Y) = \text{tr. deg}_F(L) = \text{ed}_p(\tilde{a}).$$

□
2.2. Generic elements. Let $S$ be a presheaf of sets on $\text{Var}/F$ and $X \in \text{Var}/F$. An element $a \in S(X)$ is called $p$-generic for $S$ if for any open subvariety $U \subset X$ and any $b \in S(Y)$ with the infinite field $F(Y)$ we have $b >_p (a|_U)$. Note that $F(Y)$ is infinite if either $F$ is infinite or $\dim(Y) > 0$. We say that $a$ is generic if $a$ is $p$-generic for $p = 0$. If $a$ is generic, then $a$ is $p$-generic for all $p$.

Generic elements provide an upper bound for the essential dimension.

**Proposition 2.7.** Let $S$ be a presheaf of sets on $\text{Var}/F$ and $a \in S(X)$ a $p$-generic element for $S$. Then $\text{ed}_p(S) \leq \dim(X)$.

**Proof.** Let $b \in S(Y)$. If the field $F(Y)$ is finite, we have $\text{ed}_p(b) = 0$. If $F(Y)$ is infinite, $b >_p a$ since $a$ is $p$-generic. By the definition of the essential $p$-dimension, in any case, $\text{ed}_p(b) \leq \dim(X)$, hence $\text{ed}_p(S) \leq \dim(X)$. $\square$

Clearly, if $a$ is $p$-generic, then so is the restriction $a|_U \in S(U)$ for any open subvariety $U \subset X$. This can be generalized as follows.

**Proposition 2.8.** Let $S$ be a presheaf of sets on $\text{Var}/F$, $X, Y \in \text{Var}/F$, $a \in S(X)$ and $b \in S(Y)$. Suppose that $a >_p b$ and $a$ is $p$-generic. Then $b$ is also $p$-generic for $S$.

**Proof.** Let $c \in S(Z)$ with the field $F(Z)$ infinite and $V \subset Y$ an open subvariety. Clearly, $a >_p (b|_V)$. By Lemma 2.9(1), we have $(a|_U) >_p (b|_V)$ for an open subvariety $U \subset X$. Since $a$ is $p$-generic, we have $c >_p (a|_U)$. By Lemma 2.9(1), $c >_p (b|_V)$, hence $b$ is $p$-generic. $\square$

**Theorem 2.9.** Let $S$ be a presheaf of sets on $\text{Var}/F$. If $a \in S(X)$ is a $p$-generic element for $S$, then

$$\text{ed}_p(S) = \text{ed}_p(\tilde{S}) = \text{ed}_p(\tilde{a}) = \text{ed}_p(a).$$

**Proof.** In view of Proposition 2.6, it suffices to prove that $\text{ed}_p(S) \leq \text{ed}_p(a)$. Choose an element $c \in S(Z)$ such that $a >_p c$ and $\text{ed}_p(a) = \dim(Z)$. By Lemma 2.9(1), there is an open subvariety $U \subset X$ such that $(a|_U) >_p c$.

Let $Y \in \text{Var}/F$ and let $b \in S(Y)$ be any element. If the field $F(Y)$ is finite, we have $\text{ed}_p(b) = 0$. Otherwise, as $a$ is $p$-generic, we have $b >_p (a|_U)$. It follows from Lemma 2.9(1) that $b >_p c$. Hence, in any case, $\text{ed}_p(b) \leq \dim(Z) = \text{ed}_p(a)$ and therefore, $\text{ed}_p(S) \leq \text{ed}_p(a)$. $\square$

Let $S$ be a presheaf of sets on $\text{Var}/F$. An element $a \in \tilde{S}(L)$ is called $p$-generic for $\tilde{S}$ is $a = \tilde{a}$ for a $p$-generic element $a$ for $S$.

**Example 2.10.** One can view a scheme $X$ over $F$ as a presheaf of sets on $\text{Var}/F$ by $X(Y) := \text{Mor}_F(Y, X)$ for every $Y \in \text{Var}/F$. Then the functor $\tilde{X} : \text{Fields}/F \to \text{Sets}$ coincides with the one in Proposition 2.6. It follows from Theorem 2.9 that $\text{ed}_p(X) = \dim(X)$ for all $p$.

By Proposition 2.6, for a $p$-generic element $a \in S(X)$, one has $\text{ed}_p(S) \leq \dim(X)$. The following proposition asserts that $\text{ed}_p(S)$ is equal to the dimension of a closed subvariety of $X$ with a certain property.

**Proposition 2.11.** Let $S$ be a presheaf of sets on $\text{Var}/F$ and $a \in S(X)$ a $p$-generic element for $S$. Suppose that either $F$ is infinite or $\text{ed}_p(S) > 0$. Then $\text{ed}_p(S) = \min \dim(Z)$ over all closed subvarieties $Z \subset X$ such that $a >_p (a|_Z)$. 

For any closed subvariety $Z \subset X$ with $a >_p (a|Z)$ one has $ed_p(S) = ed_p(a) \leq \dim(Z)$. We shall show that the equality holds for some $Z \subset X$.

By Corollary 4.3, there is $b \in S(Y)$ with $\dim(Y) = ed_p(a) = ed_p(S)$ and $a >_p b$. By assumption, the field $F(Y)$ is infinite. As $a$ is $p$-generic, we have $b >_p a$. By Lemma 4.7(2), there is a closed subvariety $Z \subset X$ such that $b >_p (a|Z)$. It follows that $\dim(Z) \leq \dim(Y) = ed_p(S)$. By Lemma 4.7(2), $a >_p (a|Z)$.

**Remark 2.12.** The assumption in the proposition can not be dropped (cf. Remark 4.7).

An element $a \in S(X)$ is called $p$-minimal if $ed_p(a) = \dim(X)$, i.e., whenever $\alpha >_p \beta$ for some $\beta \in S(Y)$, we have $\dim(X) \leq \dim(Y)$. By Lemma 4.7(2) and Corollary 4.3, for every $a \in S(X)$, there is a $p$-minimal $b \in S(Y)$ such that $ed_p(a) = \dim(Y)$ and $a >_p b$. It follows that $ed_p(S)$ is the maximum of $ed_p(a)$ over all $p$-minimal elements $a$.

A $p$-minimal element with $p = 0$ is called minimal.

If $a \in S(X)$ is $p$-generic $p$-minimal, then $ed_p(S) = \dim(X)$.

If $a \in S(X)$ is a $p$-generic element for $S$ and $b \in S(Y)$ is a $p$-minimal element satisfying $a >_p b$, then by Proposition 4.8, $b$ is also $p$-generic, and hence $ed_p(S) = \dim(Y)$.

The following statement gives a characterization of $p$-generic $p$-minimal elements.

**Proposition 2.13.** Let $S$ be a presheaf of sets on $\text{Var}/F$ and $a \in S(X)$ a $p$-generic element for $S$. Suppose that either $F$ is infinite or $ed_p(S) > 0$. Then $a$ is $p$-minimal if and only if for any two morphisms $f$ and $g$ from a variety $X'$ to $X$ such that $S(f)(a) = S(g)(a)$ with $f$ dominant of degree prime to $p$, the morphism $g$ is also dominant.

**Proof.** Suppose $a$ is $p$-minimal and let $f$ and $g$ be morphisms in the statement of the proposition. Let $Z$ be the closure of the image of $g$, so $a >_p (a|Z)$. By Proposition 4.11, $\dim(X) = ed_p(S) \leq \dim(Z)$, hence $Z = X$ and $g$ is dominant.

Suppose $a$ is not $p$-minimal. By Proposition 4.11, there is a proper closed subvariety $Z \subset X$ such that $a >_p (a|Z)$, i.e., there are morphisms $f : X' \to X$ and $g' : X' \to Z$ such that $S(f)(a) = S(g')(a|Z)$ and $f$ is dominant of degree prime to $p$. If $g : X' \to X$ if the composition of $g'$ with the embedding of $Z$ into $X$, then $S(f)(a) = S(g)(a)$ and $g$ is not dominant.

Specializing to the case $p = 0$ we have:

**Corollary 2.14.** In the conditions of the proposition, $a$ is minimal if and only if for any two morphisms $f$ and $g$ from a variety $X'$ to $X$ such that $S(f)(a) = S(g)(a)$ with $f$ a birational isomorphism, the morphism $g$ is dominant.

3. Essential $p$-dimension of fibered categories

The notion of the essential $p$-dimension can be defined for fibered categories over $\text{Var}/F$ or $\text{Fields}/F$ as follows (cf. [3]).

Let $\mathcal{A}$ be a category and $\varphi : \mathcal{A} \to \text{Var}/F$ a functor. For a variety $Y \in \text{Var}/F$, we write $\mathcal{A}(Y)$ for the fiber category of all objects $\xi$ in $\mathcal{A}$ with $\varphi(\xi) = Y$ and morphisms over the identity of $Y$. We assume that the category $\mathcal{A}(Y)$ is essentially small for all $Y$, i.e., the isomorphism classes of objects form a set.
Suppose that $\mathcal{A}$ is a fibered category over $\text{Var}/F$ (cf. [43]). In particular, for any morphism $f : Y \to Y'$ in $\text{Var}/F$, there is a pull-back functor $f^* : \mathcal{A}(Y') \to \mathcal{A}(Y)$ such that for any two morphisms $f : Y \to Y'$ and $g : Y' \to Y''$ in $\text{Var}/F$, the composition $f^* \circ g^*$ is isomorphic to $(g \circ f)^*$.

Let $\mathcal{A}$ be a fibered category over $\text{Var}/F$. For any $Y \in \text{Var}/F$, we write $S_\mathcal{A}(Y)$ for the set of isomorphism classes of objects in the category $\mathcal{A}(Y)$. The functor $f^*$ for a morphism $f : Y \to Y'$ in $\text{Var}/F$ induces a map of sets $S_\mathcal{A}(Y') \to S_\mathcal{A}(Y)$ making $S_\mathcal{A}$ a presheaf of sets on $\text{Var}/F$. We call $S_\mathcal{A}$ the presheaf of sets associated with $\mathcal{A}$. The essential p-dimension $\text{ed}_p(\mathcal{A})$ of $\mathcal{A}$ (respectively, the canonical p-dimension $\text{cdim}_p(\mathcal{A})$ of $\mathcal{A}$) is defined as $\text{ed}_p(S_\mathcal{A})$ (respectively, $\text{cdim}_p(S_\mathcal{A})$).

Remark 3.1. In a similar fashion, one can define the essential p-dimension for fibered categories over $\text{Fields}/F$. This notion agrees with the one given above in view of Theorem [43].

Example 3.2. Let $X$ be a scheme over $F$. Consider the category $\text{Var}/X$ of varieties over $X$, i.e., morphisms $Y \to X$ for a variety $Y$ over $F$. Morphisms in $\text{Var}/X$ are morphisms of varieties over $X$. The functor $\text{Var}/X \to \text{Var}/F$ taking $Y \to X$ to $Y$ together with the obvious pull-back functors $f^*$ make $\text{Var}/X$ a fibered category. For any variety $Y$, the fiber category over $Y$ is equal to the set $\text{Mor}_F(Y, X)$. Hence the associated presheaf of sets on $\text{Var}/F$ coincides with $X$ viewed as a presheaf as in Example [43]. It follows that $\text{ed}_p(\text{Var}/X) = \text{dim}(X)$ for all $p$.

Example 3.3. Let $G$ be an algebraic group scheme over a field $F$. The classifying space $BG$ of the group $G$ is the category with objects (right) $G$-torsors $q : E \to Y$ with $Y \in \text{Var}/F$ and morphisms between $G$-torsors $q : E \to Y$ and $q' : E' \to Y'$ given by commutative diagrams

$$
\begin{array}{ccc}
E & \longrightarrow & E' \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Y'
\end{array}
$$

with the top arrow a $G$-equivariant morphism. For every $Y \in \text{Var}/F$, the fiber category $BG(Y)$ is the category of $G$-torsors over $Y$. We write $\text{ed}_p(G)$ for $\text{ed}_p(BG)$ and call this integer the essential p-dimension of $G$. Equivalently, by Proposition [43], $\text{ed}_p(G)$ is the essential p-dimension of the functor $\text{Fields}/F \to \text{Sets}$ taking a field $L$ to the set of isomorphism classes of $G$-torsors over $L$.

Example 3.4. We can generalize the previous example as follows. Let an algebraic group scheme $G$ act on a scheme $X$ over $F$. We define the fibered category $X/G$ as follows. An object in $X/G$ over a variety $Y$ is a diagram

$$
\begin{array}{ccc}
E & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Y
\end{array}
$$

where $q$ is a $G$-torsor and $f$ is a $G$-equivariant morphism. Morphisms of diagrams in $X/G$ are defined in the obvious way. The functor $X/G \to \text{Var}/F$ takes the diagram to the scheme $Y$. The set $S_{X/G}(Y)$ consists of all isomorphism classes of
the diagrams above. For any field $L \in \text{Fields}/F$, an element of the set $\tilde{S}_{X/G}(L)$ is given by the diagram

$$
\begin{array}{c}
E' \\
\downarrow q' \\
\text{Spec } L
\end{array}
\xrightarrow{f'} X
$$

where $q'$ is a $G$-torsor and $f'$ is a $G$-equivariant morphism.

Note that if $X$ is a $G$-torsor over a scheme $Y$, then $X/G \simeq Y$, and if $X = \text{Spec } F$, then $X/G = BG$.

### 3.1. Gerbes.

Let $C$ be a commutative algebraic group scheme over $F$. There is the notion of a gerbe banded by $C$ (cf. [19, p. 144], [13, IV.3.1.1], see also examples below). There exists a bijection between the first cohomology group $H^2(F; C) := H^2_{fppf}(\text{Spec } F; C)$ and the set of isomorphism classes of gerbes banded by $C$. The trivial element in $H^2(F; C)$ corresponds to the classifying space $BC$, so $BC$ is a trivial (split) gerbe banded by $C$. In general, a gerbe banded by $C$ can be viewed as a "twisted form" of $BC$.

**Example 3.5.** Let

$$1 \to C \to G \to H \to 1$$

be an exact sequence of algebraic group schemes with $C$ a commutative group and $E \to \text{Spec } F$ an $H$-torsor. The group $G$ acts on $E$ via the map $G \to H$. The category $E/G$ is a gerbe banded by $C$. The corresponding element in $H^2(F; C)$ is the image of the class of $E$ under the connecting map

$$H^1(F, H) \to H^2(F, C).$$

**Example 3.6.** (Gerbes banded by $\mu_n$) Let $A$ be a central simple $F$-algebra and $n$ an integer with $[A] \in \text{Br}_n(F) = H^2(F, \mu_n)$. Let $X$ be the Severi-Brauer variety of $A$. Denote by $X_A$ the gerbe banded by $\mu_n$ corresponding to $[A]$. It is shown in [3] that if $n$ is a power of a prime integer $p$, then

$$\text{ed}_p(X_A) = \text{ed}(X_A) = \text{cdim}_p(X_A) + 1 = \text{cdim}(X_A) + 1 = \text{ind}(A).$$

**Example 3.7.** One can generalize the previous example as follows. Let $p$ be a prime integer and $C$ a diagonalizable algebraic group scheme of rank $s$ and exponent $p$ over $F$. In other words, $C$ is isomorphic to the product of $s$ copies of $\mu_p$. An element $\theta \in H^2(F, C)$ determines a gerbe $X$ banded by $C$. Consider the homomorphism $\beta : C^* \to \text{Br}(F)$ taking a character $\chi \in C^*$ to the image of $\theta$ under the map $H^2(F, C) \to H^2(F, G_m) = \text{Br}(F)$ induced by $\chi$. It follows from [13, 3.1] that

$$\text{ed}_p(X) = \text{ed}(X) = \text{cdim}_p(X) + s = \text{cdim}(X) + s.$$  

For a generating set $\chi_1, \chi_2, \ldots, \chi_s$ of $C^*$, let $A_1, A_2, \ldots, A_s$ be central division $F$-algebras such that $[A_i] = \beta(\chi_i)$. Set $X_i = \text{SB}(A_i)$ and $X = X_1 \times \cdots \times X_s$. Clearly, the gerbe $X$ is split over a field extension $L$ of $F$ if and only if all the algebras $A_i$ are split over $L$ if and only if $X$ has a point over $L$. It follows that $\text{cdim}_p(X) = \text{cdim}_p(X)$. 
By Example 5.4, any basis of $\text{Ker}(\beta)$ over $\mathbb{Z}/p\mathbb{Z}$ can be completed to a basis $\chi_1, \chi_2, \ldots, \chi_s$ of $C^*$ such that $X$ is $p$-minimal, i.e.,
\[ \text{cdim}_p(X) = \dim(X) = \sum_{i=1}^s (\text{ind}(A_i) - 1) = \sum_{i=1}^s (\text{ind}(\chi_i) - 1). \]
It follows from (2) that
\[ \text{ed}_p(X) = \sum_{i=1}^s \text{ind}(\chi_i). \]

4. Essential $p$-dimension of algebraic group schemes

Let $G$ be an algebraic group scheme over a field $F$. A $G$-space is a finite dimensional vector space $V$ with a (right) linear $G$-action. (Equivalently, the natural map $G \to \text{GL}(V)$ is a finite dimensional representation of $G$.) We say that $G$ acts on $V$ generically freely (or $V$ is generically free) if there is a nonempty open $G$-invariant subset $V' \subset V$ and a $G$-torsor $V' \to X$ for some scheme $X$ over $F$ (cf. [1, Def. 4.8 and 4.10]).

One can construct $G$-spaces $V$ with generically free action as follows. Embed $G$ into $\text{GL}(W)$ as a subgroup for some vector space $W$ of finite dimension and set $V = \text{End}(W)$. We view $V$ as a $G$-space via right multiplications. Then $\text{GL}(W)$ is an open $G$-invariant subset in $V$ and the natural morphism $\text{GL}(W) \to \text{GL}(W)/G$ is a $G$-torsor.

**Theorem 4.1.** (cf. [12, Lemma 6.6], [13, Example 5.4]) Let $G$ be an algebraic group scheme over a field $F$ and $V$ a $G$-space. Suppose that $G$ acts on $V$ generically freely, i.e., there is a nonempty open subset $V' \subset V$ and a $G$-torsor $a : V' \to X$ for some scheme $X$. Then the torsor $a$ is $p$-generic for all $p$.

**Proof.** Let $b : E \to Y$ be a $G$-torsor with the infinite field $F(Y)$. Let $U \subset X$ be an open subvariety. We need to show that $b >_p (a|_U)$. Replacing $X$ by $U$ and $V'$ by $a^{-1}(U)$ we may assume that $U = X$. We shall show that $b >_p a$.

The morphism $a \times b : V' \times E \to X \times Y$ is a $(G \times G)$-torsor. Considering $G$ as a diagonal subgroup of $G \times G$ we have a $G$-torsor $c : V' \times E \to Z$ and a commutative diagram
\[
\begin{array}{ccc}
V' & \longrightarrow & V' \times E \\
\downarrow a & & \downarrow b \\
X & \longleftarrow & Z \longrightarrow Y
\end{array}
\]
with the projections in the top row. The scheme $V' \times E$ is an open subset of the (trivial) vector bundle $V \times E$ over $E$. By descent, $Z$ is an open subset of a vector bundle over $Y$. Therefore, the generic fiber of $f$ is an open set of a vector space over the infinite field $F(Y)$ and hence it has a point over $F(Y)$, i.e., the generic fiber of $f$ has a splitting. It follows that there is an open subvariety $W \subset Y$ such that $f$ has a splitting $h : W \to Z$ over $W$.

Set $E' := W \times_Z (V' \times E)$. In the commutative diagram with fiber product squares
\[
\begin{array}{ccc}
E' & \longrightarrow & V' \times E \\
\downarrow & & \downarrow \\
W & \longrightarrow & Z \longrightarrow Y
\end{array}
\]
the composition in the bottom row is the inclusion morphism. Hence $E' = E|_W$ and the left vertical arrow coincides with $b|_W$. The commutative diagram

$$
\begin{array}{ccc}
E & \longrightarrow & E|_W \\
b \downarrow & & \downarrow b|_W \\
Y & \longleftarrow & W \overset{gh}{\longrightarrow} X
\end{array}
$$

then yields $b >_p a$ for all $p$.

\[ \square \]

**Corollary 4.2.** (cf. [24 Prop. 4.11]) Let $G$ be an algebraic group scheme over a field $F$. Then $e_d_p(G) \leq \dim(V) - \dim(G)$ for every generically free $G$-space $V$.

**Corollary 4.3.** Let $G$ be an algebraic group scheme over a field $F$ and $H$ a subgroup of $G$. Then $e_d_p(G) + \dim(G) \geq e_d_p(H) + \dim(H)$.

**Proof.** Let $a : V' \rightarrow X$ be the $p$-generic $G$-torsor as in Theorem 4.3. Since $H$ acts on $V$ generically freely, there is a $p$-generic $H$-torsor $b : V' \rightarrow Y$. Let $a >_p c$ for a $G$-torsor $c : E \rightarrow Z$ with $\dim(Z) = e_d_p(G)$. Let $d : E \rightarrow S$ be the $H$-torsor associated to $c$. As $a >_p c$, we have $b >_p d$ and hence

\[
e_d_p(H) \leq \dim(S) - \dim(E) - \dim(H) = \dim(Z) + \dim(G) - \dim(H) = e_d_p(G) + \dim(G) - \dim(H).
\]

\[ \square \]

**4.1. Torsion primes and special groups.** For a scheme $X$ over $F$ we let $n_X$ denote the gcd $\deg(x)$ over all closed points $x \in X$.

Let $G$ be an algebraic group scheme over $F$. A prime integer $p$ is called a torsion prime for $G$ if $p$ divides $n_E$ for a $G$-torsor $E \rightarrow \text{Spec} L$ over a field extension $L/F$ (cf. [23 Sec. 2.3]).

An algebraic group scheme $G$ over $F$ is called special if for any field extension $L/F$, every $G$-torsor over Spec $L$ is trivial. Clearly, special group schemes have no torsion primes.

The last statement of the following proposition was proven in [23 Prop. 5.3] in the case of algebraically closed field $F$.

**Proposition 4.4.** Let $G$ be an algebraic group scheme over $F$. Then a prime integer $p$ is a torsion prime for $G$ if and only if $e_d_p(G) \neq 0$. An algebraic group scheme $G$ is special if and only if $e_d(G) = 0$.

**Proof.** Let $p \geq 0$. Suppose that $p$ is not a torsion prime for $G$ if $p > 0$ or $G$ is special if $p = 0$. Let $E \rightarrow \text{Spec} L$ be a $G$-torsor over $L \in \text{Fields}/F$. As $p$ is relatively prime to $n_E$, there is a finite field extension $E'/E$ such that the $G$-torsor $E'_L$ is split and hence comes from a trivial $G$-torsor over $F$. It follows that $e_d_p(E) = 0$ and hence $e_d_p(G) = 0$.

Conversely, suppose that $e_d_p(G) = 0$ for $p \geq 0$. Assume that $F$ is infinite. Choose a $p$-minimal $p$-generic $G$-torsor $E \rightarrow X$. We claim that $n_E$ is relatively prime to $p$. Since $\dim(X) = e_d_p(G) = 0$, we have $X = \text{Spec} L$ for a finite field extension $L/F$. Let $E'$ be a trivial $G$-torsor over $F$. As $E$ is generic and the field $F$ is infinite, we have $E' >_p E$, i.e., there is a finite field extension $L'/L$ of degree prime to $p$ such that $E'_L' \cong E'_L$. Thus $E'_L'$ is trivial and hence $n_E$ is relatively prime to $p$ as $n_E$ divides $[L' : L]$.

Let $\gamma : I \rightarrow \text{Spec} K$ be a $G$-torsor over a field extension $K/F$. We need to show that $n_I$ is relatively prime to $p$. We may assume that $K \in \text{Fields}/F$. Choose
a model $c: J \to Z$ of $\gamma$, i.e., a $G$-torsor $c$ with $Z$ a model of $K$ and $\gamma$ the generic fiber of $c$. As $a$ is generic, we have $c \geq a$, i.e., a fiber product diagram

$$
\begin{array}{ccc}
J & \xleftarrow{c} & J' \xrightarrow{e} E \\
\downarrow & \downarrow & \downarrow \\
Z & \xleftarrow{f} & Z' \xrightarrow{h} X.
\end{array}
$$

with $f$ a dominant morphism of degree prime to $p$ and a $G$-torsor $c'$. Let $I' \to \text{Spec} K'$ be the generic fiber of $c'$. Since $n_I'$ divides $n_E$ and $n_E$ is relatively prime to $p$, the integer $n_I'$ is also relatively prime to $p$. It follows that $n_I$ is relatively prime to $p$ since $n_I$ divides $|K': K|n_I'$.

Now let $F$ be a finite field and $\text{ed}_p(G) = 0$. If $G$ is smooth and connected, then $G$ is special (cf. [25]). In general, if $G^o$ is the connected component of the identity and $G^o = G/G^\circ$, then the categories $BG$ and $BG^o$ are equivalent, in particular, $\text{ed}_p(G) = \text{ed}_p(G^o)$ and $G$ and $G^o$ have the same torsion primes. Thus, we may assume that $G = G^o$ is an étale group scheme. Let $K/F$ be a finite splitting field of $G$, i.e., $G_K$ is a finite constant group. Every torsion prime of $G_K$ is a torsion prime of $G$ and $\text{ed}_p(G_K) = 0$ by Proposition 4.4(1), so we may assume that $G$ is a constant group.

We claim that the order of $G$ is relatively prime to $p$. If not, let $H$ be a finite subgroup of $G$ of order $p$ if $p > 0$ and of any prime order if $p = 0$. We have $\text{ed}_p(G) \geq \text{ed}_p(H) > 0$ by Corollary 4.8, a contradiction. Thus, $|G|$ is relatively prime to $p$. Then every $G$-torsor $E$ (a Galois $G$-algebra) is split by a finite field extension of degree prime to $p$, i.e., $n_E$ is relatively prime to $p$ and $p$ is not a torsion prime of $G$.

**Theorem 4.5.** Let $G$ be an algebraic group scheme. Assume that either $G$ is not special or $F$ is infinite. Let $a: E \to X$ be a generic $G$-torsor and let $d$ be the smallest dimension of the image of a rational $G$-equivariant morphism $E \to E$. Then $\text{ed}(G) = d - \dim(G)$.

**Proof.** Let $f: E \to E$ be a rational $G$-equivariant morphism. Denote by $f': X' \to X$ the corresponding rational morphism. Let $Z$ be the closure of the image of $f$, so dimension of the image of $f$ is equal to $\dim(Z) + \dim(G)$. There are morphisms $g: X' \to X$ and $h: X' \to Z$ with $g$ a birational isomorphism such that $g^*(E) \simeq h^*(E|_Z)$, i.e., $a > (a|_Z)$. The statement of the theorem follows now from Proposition 4.11. 

**Corollary 4.6.** Let $G$ be an algebraic group scheme. Assume that either $G$ is not special or $F$ is infinite. Let $a: E \to X$ be a generic $G$-torsor. Then $a$ is minimal if and only if every rational $G$-equivariant morphism $E \to E$ is dominant.

**Remark 4.7.** Corollary 4.6 fails for special groups over a finite field. Indeed, let $G$ be the trivial group over a finite field and let $X$ be the affine line with all rational points removed. Since $X$ has no rational points, every rational morphism $X \to X$ is dominant. But the identity morphism of $X$, which is obviously a generic $G$-torsor, is not a minimal $G$-torsor as $\text{ed}(G) = 0$.

**4.2. A lower bound.** The following statement was proven in [3].

**Theorem 4.8.** Let $f: G \to H$ be a homomorphism of algebraic group schemes. Then for any $H$-torsor $E$ over $F$, we have $\text{ed}_p(G) \geq \text{ed}_p(E/G) - \dim(H)$. 
Proof. Let $L/F$ be a field extension and let $x = (J, g, \alpha)$ be an object of $E/G$ over Spec$(L)$. Let $\beta : f_*(J) \to E$ be the isomorphism of $H$-torsors induced by $\alpha$. Choose a field extension $L'/L$ of degree prime to $p$ and a subfield $K \subset L'$ over $F$ such that $\text{tr.deg}_F(K) = \text{ed}_p(J)$ and there is a $G$-torsor $I$ over $K$ with $I_L \cong J_{L'}$.

We shall write $Z$ for the scheme of isomorphisms $\text{Iso}_K(f_*(J), E_K)$ of $H$-torsors over $K$. Clearly, $Z$ is a torsor over $K$ for the twisted form $\text{Aut}_K(f_*(J))$ of $H$, so $\dim_K(Z) = \dim(H)$. The image of the morphism Spec $L' \to Z$ over $K$ representing the isomorphism $\beta_{L'}$ is a one-point set $\{z\}$ of $Z$. Therefore, $\beta_{L'}$ and hence $x_{L'}$ are defined over $K(z)$. It follows that

$$\text{ed}_p(J) + \dim(H) = \text{tr.deg}_F(K) + \dim_K(Z) \geq \text{tr.deg}_F(K(z)) \geq \text{ed}_p(x).$$

Hence

$$\text{ed}_p(G) \geq \text{ed}_p(J) \geq \text{ed}_p(x) - \dim(H),$$

and $\text{ed}_p(G) \geq \text{ed}_p(E/G) - \dim(H)$. \qed

4.3. Essential dimension of spinor groups. Let $\text{Spin}_n$, $n \geq 3$, be the split spinor group over a field of characteristic 2. The following inequalities are proved in [3] Th. 3.3 if $n \geq 15$:

- $\text{ed}_2(\text{Spin}_n) \geq 2^{(n-1)/2} - n(n-1)/2$ if $n$ is odd
- $\text{ed}_2(\text{Spin}_n) \geq 2^{(n-2)/2} - n(n-1)/2$ if $n \equiv 2 \pmod{4}$
- $\text{ed}_2(\text{Spin}_n) \geq 2^{(n-2)/2} + 1 - n(n-1)/2$ if $n \equiv 0 \pmod{4}$

Moreover, if $\text{char}(F) = 0$, then

- $\text{ed}_2(\text{Spin}_n) = \text{ed}(\text{Spin}_n) = 2^{(n-1)/2} - n(n-1)/2$ if $n$ is odd
- $\text{ed}_2(\text{Spin}_n) = \text{ed}(\text{Spin}_n) = 2^{(n-2)/2} - n(n-1)/2$ if $n \equiv 2 \pmod{4}$
- $\text{ed}_2(\text{Spin}_n) \leq \text{ed}(\text{Spin}_n) \leq 2^{(n-2)/2} + n(n-1)/2$ if $n \equiv 0 \pmod{4}$

We improve the lower bound for $\text{ed}_2(\text{Spin}_n)$ in the case $n \equiv 0 \pmod{4}$.

Theorem 4.9. Let $n$ be a positive integer divisible by 4 and $\text{Spin}_n$ the split spinor group over a field $F$ of characteristic different from 2. Let $2^k$ be the largest power of 2 dividing $n$. Then

$$\text{ed}_2(\text{Spin}_n) \geq 2^{(n-2)/2} + 2^k - n(n-1)/2.$$ 

Proof. The center $C$ of the group $G = \text{Spin}_n$ is isomorphic to $\mu_2 \times \mu_2$. The factor group $H = G/C$ is the special projective orthogonal group (cf. [3]). An $H$-torsor over a field extension $L/F$ determines a central simple algebra $A$ with an orthogonal involution $\sigma$ of trivial discriminant. The image of the map $C^* \to \text{Br}(L)$ is equal to $\{0, [A], [C^+], [C^-]\}$, where $C^+$ and $C^-$ are simple components of the Clifford algebra $C(A, \sigma)$. By [3], there is a field extension $L/F$ and an $H$-torsor $E$ over $L$ such that $\text{ind}(C^+) = \text{ind}(C^-) = 2^{(n-2)/2}$ and $\text{ind}(A) = 2^k$, the largest power of 2 dividing $n$. By Example 4.4,

$$\text{ed}_2(E/G) = \text{ind}(A) + \text{ind}(C^+) = 2^{(n-2)/2} + 2^k.$$ 

It follows from Theorem [3] that

$$\text{ed}_2(\text{Spin}_n) \geq \text{ed}_2(E/G) - \dim(H) = 2^{(n-2)/2} + 2^k - n(n-1)/2. \quad \Box$$
COROLLARY 4.10. If $n$ is a power of 2 and $\text{char}(F) = 0$ then
\[
\text{ed}_2(\text{Spin}_n) = \text{ed}(\text{Spin}_n) = 2^{(n-2)/2} + n - n(n-1)/2.
\]

Below is the table of values $d_n := \text{ed}_2(\text{Spin}_n) = \text{ed}(\text{Spin}_n)$ over a field of characteristic zero (cf. [8]):

<table>
<thead>
<tr>
<th>$n$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_n$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>23</td>
<td>24</td>
<td>120</td>
<td>103</td>
<td>341</td>
<td></td>
</tr>
</tbody>
</table>

The torsors for $\text{Spin}_n$ are essentially the isomorphism classes of quadratic forms in $I^3$, where $I$ is the fundamental ideal in the Witt ring of $F$. A jump of the value of $\text{ed}(\text{Spin}_n)$ when $n > 14$ is probably related to the fact that there is no simple classification of quadratic forms in $I^3$ of dimension greater than 14.

5. Essential $p$-dimension of finite groups

Let $G$ be a finite group. We consider $G$ as a constant algebraic group over a field $F$. A $G$-torsor $E$ over $\text{Spec}(L)$ for a field extension $L/F$ is of the form $E = \text{Spec}(A)$, where $A$ is a Galois $G$-algebra over $L$. Thus, the fibered category $\text{BG}$ is equivalent to the category of Galois $G$-algebras over field extensions of $F$.

A generically free $G$-space is the same as a faithful $G$-space, i.e., a $G$-space $V$ such that the group homomorphism $G \to \text{GL}(V)$ is injective. By Corollary 4.3, $\text{ed}(G) \leq \dim(V)$ for any faithful $G$-space $V$. The essential dimension $\text{ed}(G)$ can be smaller than dimension of every any faithful $G$-space $V$. For example, for the symmetric group $S_n$ one has $\text{ed}(S_n) \leq n - 2$ if $n \geq 3$ (cf. [8, Th. 6.5]), whereas the least dimension of a faithful $S_n$-space is equal to $n - 1$. Note that the value of $\text{ed}(S_n)$ is unknown for $n \geq 7$.

Computation of the essential $p$-dimension of a finite group $G$ for $p > 0$ is somewhat simpler. The following proposition shows that $G$ can be replaced by a Sylow $p$-subgroup.

PROPOSITION 5.1. Let $G$ be a finite group and $H \subset G$ a Sylow $p$-subgroup. Then $\text{ed}_p(G) = \text{ed}_p(H)$.

PROOF. By Corollary [4, 8], $\text{ed}_p(G) \geq \text{ed}_p(H)$. Let $A$ be a Galois $G$-algebra over a field $L \in \text{Fields}/F$. Then the subalgebra $A^H$ of $H$-invariant elements is an étale $L$-algebra of rank prime to $p$. Let $e \in A^H$ be an idempotent such that $K = A^H e$ is a field extension of $L$ of degree prime to $p$. Then $Ae$ is a Galois $H$-algebra over $K$. Choose a field extension $K'/K$ of degree prime to $p$ and a subfield $M \subset K$ over $F$ such that there is a Galois $H$-algebra $B$ over $M$ with $B \otimes_M K' \simeq Ae \otimes_K K'$ and $\text{ed}_p(Ae) = \text{tr.deg}_F(M) \leq \text{ed}_p(H)$.

For any Galois $H$-algebra $C$ we write $\mathcal{C}$ for the algebra $\text{Map}_H(G, C)$ of $H$-equivariant maps $G \to C$. Clearly, $\mathcal{C}$ has structure of a Galois $G$-algebra. Considering $A$ as a Galois $H$-algebra over $A^H$, we have an isomorphism of Galois $G$-algebras
\[
A \otimes_L (A^H) \to \overline{A}
\]

taking $a \otimes a'$ to the map $f : G \to A$ defined by $f(g) = g(a)a'$. It follows that
\[
\mathcal{C} \otimes_M K \simeq \overline{\mathcal{C}} \otimes_K K' \simeq \overline{\mathcal{C}} \otimes_K K' \simeq A \otimes_L (A^H e) \otimes_K K' = A \otimes_L K'.
\]

Hence, $A$ is $p$-defined over $M$ and the essential $p$-dimension of the Galois $G$-algebra $A$ is at most $\text{tr.deg}_F(M) \leq \text{ed}_p(H)$. It follows that $\text{ed}_p(G) \leq \text{ed}_p(H)$. \qed
By Proposition [23, (2)], the integer \( \text{ed}_p(G) \) does not change under field extensions of \( F \) of degree prime to \( p \). It follows then from Proposition [23] that \( \text{ed}_p(G) \leq \dim(V) \) for any faithful \( H \)-space \( V \) for a Sylow \( p \)-subgroup \( H \) of \( G \) over the field \( F(\xi_p) \), where \( \xi_p \) is a primitive \( p \)-th root of unity.

The following statement was proven in [32, Th. 4.1, Rem. 4.8].

**Theorem 5.2.** Let \( p \) be a prime integer and let \( F \) be a field of characteristic different from \( p \). Then the essential \( p \)-dimension \( \text{ed}_p(G) \) over \( F \) of a finite group \( G \) is equal to the least dimension of a faithful \( H \)-space of a Sylow \( p \)-subgroup \( H \) of \( G \) over the field \( F(\xi_p) \).

**Proof.** By Propositions [23] and [23] we may assume that \( G \) is a \( p \)-group and \( F \) contains a primitive \( p \)-th root of unity.

By Corollary [23] it suffices to find a faithful \( G \)-space \( V \) with \( \text{ed}_p(G) \geq \dim(V) \).

Denote by \( C \) the subgroup of all central elements of \( G \) of exponent \( p \) and set \( H = G/C \), so we have an exact sequence

\[
1 \to C \to G \to H \to 1.
\]

Let \( E = \text{Spec } F \) be an \( H \)-torsor over \( F \) and let \( C^* \) denote the character group \( \text{Hom}(C, G_m) \) of \( C \). The \( H \)-torsor \( E \) over \( F \) yields the homomorphism

\[
\beta^E : C^* \to \text{Br}(F)
\]

taking a character \( \chi : C \to G_m \) to the image of the class of \( E \) under the composition

\[
H^1(F, H) \xrightarrow{\partial} H^2(F, C) \xrightarrow{\chi} H^2(F, G_m) = \text{Br}(F),
\]

where \( \partial \) is the connecting map for the exact sequence (3). Note that as \( \mu_p \subset F^\times \), we can identify \( C \) with \( (\mu_p)^s \), i.e., \( C \) is a diagonalizable group of exponent \( p \).

Consider the gerbe \( E/G \) banded by \( C \). The class of \( E/G \) in \( H^2(F, C) \) coincides with the image of the class of \( E \) under \( \partial \).

By Example [23], there is a basis \( \chi_1, \chi_2, \ldots, \chi_s \) of \( C^* \) such that

\[
\text{ed}_p(E/G) = \sum_{i=1}^s \text{ind} \beta^E(\chi_i).
\]

Now we choose a specific \( E \), namely a generic \( H \)-torsor over a field extension \( L \) of \( F \). Let \( \chi : C \to G_m \) be a character and \( \text{Rep}^{(\chi)}(G) \) the category of all \( G \)-spaces such that \( \psi = \chi(c)v \) for any \( c \in C \) and \( v \in V \). By Theorem [23],

\[
\text{ind} \beta^E(\chi) = \gcd \dim(V)
\]

over all \( G \)-spaces \( V \) in \( \text{Rep}^{(\chi)}(G) \). Note that dimension of every irreducible \( G \)-space is a power of \( p \). Indeed, let \( q \) be the order of \( G \). By [23, Th. 24], every irreducible \( G \)-space is defined over the field \( K = F(\mu_q) \). Since \( F \) contains \( p \)-th roots of unity, the degree \( [K : F] \) is a power of \( p \). Let \( V \) be an irreducible \( G \)-space over \( F \). Write \( V_K \) as a direct sum of irreducible \( G \)-spaces \( V_j \) over \( K \). As each \( V_j \) is absolutely irreducible, \( \dim(V_j) \) divides \( q \) and hence \( \dim(V_j) \) is a power of \( p \). The group \( \Gamma = \text{Gal}(K/F) \) permutes transitively the \( V_j \). As \( |\Gamma| \) is a power of \( p \), the number of the \( V_j \)'s is also a power of \( p \).

Hence, the \( \gcd \) in (3) can be replaced by \( \min \). Therefore, for any character \( \chi \in C^* \), there is a \( G \)-space \( V_\chi \in \text{Rep}^{(\chi)}(G) \) such that \( \text{ind} \beta^E(\chi) = \dim(V_\chi) \). Let \( V \) be the direct sum of the \( V_\chi \)'s for \( i = 1, \ldots, s \). It follows from (5) that

\[
\text{ed}_p(E/G) = \dim(V).
\]
Applying Proposition 4.3(1) and Theorem 4.8 for the gerbe $E/G$ over the field $L$, we get the inequality

$$\text{ed}_p(G) \geq \text{ed}_p(G_+) \geq \text{ed}_p(E/G) = \dim(V).$$

It suffices to show that $V$ is a faithful $G$-space. Since the $\chi_i$ form a basis of $C^*$, the $C$-space $V$ is faithful. Let $N$ be the kernel of $V$. We have $N \cap C = \{1\}$. As every nontrivial normal subgroup of $G$ intersects $C$ nontrivially, it follows that $N = \{1\}$, i.e., the $G$-space $V$ is faithful.

**Corollary 5.3.** Let $G$ be a $p$-group and let $F$ be a field containing $p$-th roots of unity. Then $\text{ed}(G)$ coincides with $\text{ed}_p(G)$ and is equal to the least dimension of a faithful $G$-space over $F$.

**Proof.** Let $V$ be a faithful $G$-space of the least dimension. Then by Theorem 5.4 and Corollary 4.8,

$$\dim(V) = \text{ed}_p(G) \leq \text{ed}(G) \leq \dim(V).$$

The case of a cyclic group was considered in [10]:

**Corollary 5.4.** Let $G$ be a cyclic group of a primary order $p^n$ and let $F$ be a field containing $p$-th roots of unity. Then $\text{ed}(G) = \text{ed}_p(G) = [F(\xi_{p^n}) : F]$.

**Proof.** The $G$-space $F(\xi_{p^n})$ with a generator of $G$ acting by multiplication by $\xi_{p^n}$ is a faithful irreducible $G$-space of the least dimension.

## 6. Appendix

### 6.1. Models.

For any $X \in \text{Var}/F$, the field $F(X)$ lies in $\text{Fields}/F$. Conversely, let $L \in \text{Fields}/F$. A model of $L$ is a variety $X \in \text{Var}/F$ together with an isomorphism $F(X) \cong L$ over $F$. A morphism of two models $X$ and $X'$ of $L$ is a (unique) birational isomorphism between $X$ and $X'$ preserving the identifications of the field $F(X)$ and $F(X')$ with $L$.

Let $K \subset L$ be a subfield and $Y$ a model of $K$, so we have a morphism $\text{Spec} L \to Y$. Then there is a model $X$ of $L$ and a dominant morphism $f : X \to Y$ inducing the field embedding $K \hookrightarrow L$. Indeed, we can start with any model $X$ of $L$ and then replace it by the graph of the corresponding rational morphism $X \dashrightarrow Y$. The morphism $f$ is called a model of the morphism $\text{Spec} L \to Y$.

Let $p$ be a prime integer.

**Lemma 6.1 (cf. [13, Lemma 3.3]).** Let $K$ be an arbitrary field, $K'/K$ a finite field extension of degree prime to $p$, and $K \to L$ a field homomorphism. Then there exists a field extension $L'/L$ of degree prime to $p$ and a field homomorphism $K' \to L'$ extending $K \to L$.

**Proof.** We may assume that $K'$ is generated over $K$ by one element. Let $f(t) \in F[t]$ be its minimal polynomial. Since the degree of $f$ is prime to $p$, there exists an irreducible divisor $g \in L[t]$ of $f$ over $L$ such that $\deg(g)$ is prime to $p$. We set $L' = L[t]/(g)$.

**Lemma 6.2.** Let $f : X' \to X$ be a morphism of varieties over $F$ of degree prime to $p$. Then there is an open subvariety $U \subset X$ such that for every $x \in U$ there exists a point $x' \in X'$ with $f(x') = x$ and the degree $[F(x') : F(x)]$ prime to $p$. 


Proof. Let $U \subset X$ be an open subvariety such that the restriction $f^{-1}(U) \to U$ of $f$ is flat of degree $d$ (prime to $p$). Then for every $x \in U$, the fiber $f^{-1}(x)$ is a finite scheme over $F(x)$ of degree $d$, i.e., $f^{-1}(x) = \text{Spec} A$ for an $F(x)$-algebra $A$ of dimension $d$. The artinian ring $A$ is a product of local rings $A_i$ with maximal ideals $P_i$. We have

$$d = \sum \dim(A_i) = \sum \dim(A_i/P_i) \cdot l(A_i),$$

where $l(A_i)$ is the length of the $A$-module $A_i$ and dimension is taken over $F(x)$. As $d$ is prime to $p$, there is an $i$ such that $\dim(A_i/P_i)$ is prime to $p$. The corresponding point $x' \in f^{-1}(x)$ satisfies the required conditions.

Lemma 6.3. Let $g : X \to Y$ and $h : Y' \to Y$ be morphisms of varieties over $F$. Let $y \in Y$ be the image of the generic point of $X$. Suppose that there is a point $y' \in Y'$ such that $h(y') = y$ and $[F(y') : F(y)]$ is prime to $p$. Then there exists a commutative square of morphisms of varieties

$$
\begin{array}{ccc}
X' & \to & X \\
\downarrow m & & \downarrow g \\
Y' & \to & Y
\end{array}
$$

with $m$ dominant of degree prime to $p$.

Proof. We view the residue field $F(y)$ as a subfield of the fields $F(X)$ and $F(y')$. By Lemma [13], there is a field extension $L$ of $F(X)$ and $F(y')$ such that $[L : F(X)]$ is prime to $p$. The natural morphisms $\text{Spec} L \to X$ and $\text{Spec} L \to Y'$ yield a morphism $\text{Spec} L \to X \times_Y Y'$. Clearly, a model $X' \to X \times_Y Y'$ of this morphism together with the projections $m : X' \to X$ and $l : X' \to Y'$ fit in the required diagram.

6.2. Valuations and places. A geometric valuation of a field $L \in \text{Fields}/F$ is a valuation $v$ of $L$ over $F$ with residue field $F(v)$ such that $\text{rank}(v) = \text{tr. deg}_F(L) - \text{tr. deg}_F F(v)$. The residue field of a geometric valuation is necessarily finitely generated over $F$ (cf. [21]).

Let $L$ and $K$ be field extensions of $F$. An $F$-place $\pi : K \to L$ is a local ring homomorphism $R \to K$ of a valuation ring $R$ in $L$ containing $F$. The ring $R$ is called the valuation ring of $\pi$. We say that $\pi$ is geometric is the valuation of $R$ is geometric.

If $\pi : L \to K$ and $\rho : M \to L$ are two places, then the composition of places $\pi \circ \rho : M \to K$ is defined. If $\pi$ and $\rho$ are geometric, then so is $\pi \circ \rho$.

A geometric place is a composition of places with discrete geometric valuation rings.

Lemma 6.4. Let $L \in \text{Fields}/F$, let $v$ be a geometric valuation of $L$ over $F$ and let $L'/L$ be a finite field extension of degree prime to $p$. Then there exists a geometric valuation $v'$ of $L'$ extending $v$ such that the degree of the residue field extension $F(v')/F(v)$ is prime to $p$.

Proof. If $L'/L$ is separable and $v_1, \ldots, v_k$ are all the extensions of $v$ on $L'$, then $[L' : L] = \sum e_i [F(v_i) : F(v)]$ where $e_i$ is the ramification index (cf. [21], Ch. VI, Th. 20 and p. 63). It follows that the integer $[F(v_i) : F(v)]$ is prime to $p$ for some $i$. 

Choose a regular system of parameters. Let \( L \) be the generic fiber of the field \( K \) places. We set \( F \) valuation ring with quotient field \( F \). Set \( F(X) = F_0 \to F_1 \to \cdots \to F_n = F(x) \) is a geometric valuation satisfying the required conditions.

Proof. Choose a regular system of parameters \( a_1, a_2, \ldots, a_n \) in the regular local ring \( R = O_{X,F} \). Let \( M_i \) be the ideal of \( R \) generated by \( a_1, \ldots, a_n \). Set \( R_i = R/M_i \) and \( P_i = M_{i+1}/M_i \). Denote by \( F_i \) the quotient field of \( R_i \); in particular, \( F_0 = F(X) \) and \( F_n = F(x) \). The localization ring \( (R_i)_{P_i} \) is a discrete geometric valuation ring with quotient field \( F_i \) and residue field \( F_{i+1} \), therefore it determines a geometric place \( F_i \to F_{i+1} \). The valuation corresponding to the composition of places

\[
F(X) = F_0 \to F_1 \to \cdots \to F_n = F(x)
\]

is a geometric valuation satisfying the required conditions.

6.3. Indices of algebras. Let \( G \) be a finite group and \( C \) a central subgroup. We set \( H = G/C \). Let \( W \) be a faithful \( H \)-space and \( W' \) an open subset of the affine space of \( W \) where \( H \) acts freely, so that there is an \( H \)-torsor \( \pi : W' \to Y \). Let \( E \) be the generic fiber of the \( H \)-torsor \( \pi \). It is a generic \( H \)-torsor over the function field \( L = F(Y) \). Consider the homomorphism \( \beta^E : C^* \to Br(F) \) defined in (4).

Let \( \chi : C \to G_m \) be a character and let \( \operatorname{Rep}(\chi)(G) \) be the category of all \( G \)-spaces such that \( v^c = \chi(c)v \) any \( c \in C \) and \( v \in V \).

Theorem 6.7. Let \( G \) be a finite group and let \( C \) be a central subgroup of \( G \). Assume that \( |C| \) is not divisible by \( \operatorname{char} F \). Set \( H = G/C \) and let \( E \) be a generic \( H \)-torsor. Then for any character \( \chi \in C^* \), we have \( \operatorname{ind}_{H} \beta^E(\chi) = \gcd \dim(V) \) over all \( G \)-spaces \( V \) in \( \operatorname{Rep}(\chi)(G) \).

In the rest of the section we give a proof of this theorem.

Let \( S \) be a commutative ring and \( H \) a finite group acting on \( S \) (on the right) by ring automorphisms. Set

\[
R = S^H := \{ s \in S \text{ such that } s^h = s \text{ for all } h \in H \}
\]

and denote by \( S \star H \) the crossed product with trivial factors. Precisely, \( S \star H \) consists of formal sums \( \sum_{h \in H} h s_h \) with \( s_h \in S \). The product is given by the rule \( (hs)(h't') = (hh')(s^h t') \).

Let \( M \) be a (right) \( S \)-module. Suppose that \( H \) acts on \( M \) on the right such that \((ms)^h = m^h s^h\). Then \( M \) is a right \( S \star H \)-module by \( m(hs) = m^h s \). Conversely, a right \( S \star H \)-module is a right \( S \)-module together with a right \( H \)-action as above. If \( M \) is a right \( S \star H \)-module, then the subset \( M^H \) of \( H \)-invariant elements in \( M \) is an \( R \)-module. We have a natural \( S \)-module homomorphism \( M^H \otimes_R S \to M \), \( m \otimes s \mapsto ms \).
We say that $S$ is a Galois $H$-algebra over $R$ is the morphism Spec $S \to$ Spec $R$ is an $H$-torsor.

**Proposition 6.8.** (cf. [M]) The following are equivalent:

1. $S$ is a Galois $H$-algebra over $R$.
2. The morphism Spec $S \to$ Spec $R$ is an $H$-torsor.
3. For any $h \in H$, $h \neq 1$, the elements $s^h - s$ with $s \in S$ generate the unit ideal in $S$.
4. For every left $S \ast H$-module $M$, the natural map $M^H \otimes_R S \to M$ is an isomorphism.

**Corollary 6.9.** Let $S$ be an Galois $H$-algebra over $R$. Then the functors between the categories of finitely generated right modules

$$M(R) \to M(S \ast H), \quad N \mapsto N \otimes_R S$$

$$M(S \ast H) \to M(R), \quad M \mapsto M^H$$

are equivalences inverse to each other.

**Proof of Theorem 6.8.** Let $W$ be a faithful $H$-space. Let $S$ denote the symmetric algebra of the dual space $W^*$. The group $H$ acts on $S$. Set $R = S^H$, $Y = \text{Spec}(R)$ and $L = F(Y)$ the quotient field of $R$.

For any $h \in H$, $h \neq 1$, there is a linear form $\varphi_h \in W^*$ satisfying $(\varphi_h)^h \neq \varphi_h$. Set

$$r = \prod_{h,h' \in H, h \neq 1} ((\varphi_h)^{hh'} - (\varphi_h)^{h'})$$

in $S$. We have $r \in R$ and $r \neq 0$. For any $h \neq 1$, the element $(\varphi_h)^h - \varphi_h$ is invertible in the localization ring $S_r$. By Proposition 6.8, the localization ring $S_r$ is a Galois $H$-algebra over $R_r$.

Let $\chi : C \to \mathbf{G}_m$ be a character of $C$. Note that $G$ acts upon $S$ via the group homomorphism $G \to H$, so we have the ring $S \ast G$ defined. We write $M^{(\chi)}(S \ast G)$ for the full subcategory of $M(S \ast G)$ consisting of all modules $M$ with $m^c = \chi(c)m$ for all $m \in M$ and $c \in C$. We also write $K_0^{(\chi)}(S \ast G)$ for the Grothendieck group of $M^{(\chi)}(S \ast G)$. Note that $K_0^{(\chi)}(S \ast G)$ is a natural direct summand of $K_0(S \ast G)$.

Fix a $G$-space $U \in \text{Rep}^{(\chi)}(G)$ and set $U_{S_r} = U \otimes_F S_r$. We have

$$\text{End}(U) \otimes_R S_r \simeq \text{End}_{S_r}(U_{S_r}).$$

The conjugation $G$-action on $\text{End}(U)$ factors through an $H$-action. Consider the algebra $A = \text{End}_{S_r}(U_{S_r})^H$ over $R_r$. By Proposition 6.8(4),

$$A \otimes_{R_r} S_r \simeq \text{End}_{S_r}(U_{S_r}),$$

hence $A$ is an Azumaya $R_r$-algebra (by descent, as $S_r$ is a faithfully flat $R_r$-algebra).

Recall that $L = F(Y)$ is the quotient field of $R$. Set

$$A = A \otimes_{R_r} L.$$

Clearly, $A$ is a central simple algebra over $L$ of degree $\dim U$. We also have

$$A = (\text{End}(U) \otimes_F L')^H,$$

where $L'$ is the quotient field of $S$. Moreover, $[A] = \beta_E(\chi)$ in $\text{Br}(L)$.

The localization in algebraic $K$-theory provides a surjective homomorphism

$$(7) \quad K_0(A) \to K_0(A).$$
By Corollary 6.10, the category of right $A$-modules and right $\text{End}_{S_r}(U_{S_r}) \star H$-modules are equivalent. Thus the functor $M \mapsto M^H$ induces an isomorphism

$$K_0(\text{End}_{S_r}(U_{S_r}) \star H) \sim K_0(A).$$

The category of right $\text{End}_{S_r}(U_{S_r}) \star H$-modules is equivalent to the subcategory of right $\text{End}_{S_r}(U_{S_r}) \star G$-modules with the group $G$ acting trivially. Hence we have an isomorphism

$$K_0^{(1)}(\text{End}_{S_r}(U_{S_r}) \star G) \sim K_0(\text{End}_{S_r}(U_{S_r}) \star H).$$

By Morita equivalence, the functors

$$M(S_r \star G) \rightarrow M(\text{End}_{S_r}(U_{S_r}) \star G), \quad N \mapsto N \otimes_F U^*$$

$$M(\text{End}_{S_r}(U_{S_r}) \star G) \rightarrow M(S_r \star G), \quad M \mapsto M \otimes_{\text{End}(U)} U$$

are equivalences inverse to each other. Moreover, under these equivalences, the subcategory $M^{(\chi)}(S_r \star G)$ corresponds to $M^{(1)}(\text{End}_{S_r}(U_{S_r}) \star G)$. Hence we get an isomorphism

$$K_0^{(\chi)}(S_r \star G) \sim K_0^{(1)}(\text{End}_{S_r}(U_{S_r}) \star G).$$

By localization, we have a surjection

$$K_0^{(\chi)}(S \star G) \rightarrow K_0^{(\chi)}(S_r \star G).$$

The ring $S$ is graded with $S_0 = F$. We view the ring $B = S \star G$ as a graded ring with $B_0 = F \star G = FG$ (the group algebra). Note that $B$ is a free left $B_0$-module. As the global dimension of the polynomial ring $S$ is finite, we can choose a finite projective resolution $P^* \rightarrow F$ of the $S$-module $F = S_0$. Since $B$ is a free right $S$-module, $B \otimes_S P^* \rightarrow B \otimes_S F$ is a finite projective resolution of the left $B$-module $B \otimes_S F = FG = B_0$. Hence $B_0$ has finite Tor-dimension as a left $B$-module.

Therefore, $B$ satisfies the conditions of the following theorem:

**Theorem 6.10.** [20], Th. 7] Let $B = \prod_{i \geq 0} B_i$ be a graded Noetherian ring. Suppose:

1. $B$ is flat as a left $B_0$-module,
2. $B_0$ is of finite Tor-dimension as a left $B$-module.

Then the exact functor $M(B_0) \rightarrow M(B)$ taking an $M$ to $M \otimes_{B_0} B$ yields an isomorphism

$$K_0(B_0) \sim K_0(B).$$

By Theorem 6.10 applied to the graded ring $B = S \star G$, there is a canonical isomorphism

$$K_0(\text{Rep}(G)) = K_0(FG) = K_0(B_0) \sim K_0(B) = K_0(S \star G).$$

Moreover, this isomorphism takes $K_0(\text{Rep}^{(\chi)}(G))$ onto $K_0^{(\chi)}(S \star G)$, i.e., we have an isomorphism

$$K_0(\text{Rep}^{(\chi)}(G)) \sim K_0^{(\chi)}(S \star G).$$

The surjective composition $K_0(\text{Rep}^{(\chi)}(G)) \rightarrow K_0(A)$ of the surjective maps (4)-(12) takes the class of a $G$-space $V \in \text{Rep}^{(\chi)}(G)$ to the class of the right $A$-module

$$(V \otimes_F U^* \otimes_F L')^H$$
of dimension \( \dim(V) \cdot \dim(U) \) over the field \( L \). On the other hand, the group \( K_0(A) \) is infinite cyclic group generated by the class of a simple module of dimension \( \text{ind}(A) \cdot \dim(U) \) over \( L \). The result follows.

References


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