THE GROUP OF K₁-ZERO-CYCLES ON SEVERI-BRAUER VARIETIES

A. S. Merkurjev and A. A. Suslin
St. Petersburg Department of Mathematical Institute
St. Petersburg 198904, Russia

For any algebraic variety $X$ of dimension $d$ over a field $F$, one can define the following complex [7]:

$$
\bigcup_{x \in X^0} K_n F(x) \rightarrow \bigcup_{x \in X^1} K_{n-1} F(x) \rightarrow \cdots \rightarrow \bigcup_{x \in X^d} K_{n-d} F(x)
$$

where $X^i$ is the set of points of codimension $i$ in $X$. Cohomology groups of this complex we'll denote by $H^i(X,K_n)$ and call $K$-cohomology groups. In particular, the group $H^i(X,K_i)$ coincides with the Chow group of the cycles of codimension $i$ [7]. The group of $K_n$-zero-cycles $H^d(X,K_{n+d})$ we'll denote by $H_0(X,K_n)$.

Let $X$ be a Severi-Brauer variety associated with a central simple $F$-algebra $D$ [2]. In the case when the index of $D$ is a prime number, some $K$-cohomology groups were computed in [3]. The group of zero-cycles $H_0(X,K_0)$ was computed in [5] for any Severi-Brauer variety $X$. The present paper is devoted to the computation of the group $H_0(X,K_1)$ also for any Severi-Brauer variety $X$.

For any $n \geq 0$ we construct a homomorphism

$$p_n: H_0(X,K_n) \rightarrow K_n D.$$

The result of Panin mentioned above shows that $p_0$ is an isomorphism. It is not difficult to show that for $n \geq 3$ in general $p_n$ is neither injective nor surjective. The main result of the present paper is the proof of bijectivity of $p_1$. It seems reasonable
that $p_2$ is also always an isomorphism. (At least this is true for one-dimensional Severi-Brauer varieties).

The paper is organized as follows. In the first section the technique of specialization is developed. In Section 2 we define the homomorphism $p_n$ (the definition of $p_0$ and $p_1$ is possible without using the higher algebraic $K$-theory). The rest of the paper is devoted to the construction of the inverse map to $p_1$ which at first is defined with help of the technique of specialization of some "dense" subset (Section 3) and then is extended to the whole group $K_1 D$.

Some words about notation. If $X$ is a variety over a field $F$, $D$ is any $F$-algebra then for any commutative $F$-algebra $B$ we write:

$$X_B = X \times \text{Spec}B, D_B = D \otimes_F B.$$

1. Specialization

In this section we develop the technique which will be used in consequence. Let $X$ be an algebraic variety over a field $F$, $R$ be an $F$-algebra which is a discrete valuation ring with residue field $k$ and fraction field $K$, $\pi \in R$ be any prime element, and $D$ be a central simple $F$-algebra. We construct the homomorphisms of specialization in the following three situations:

1. The category of coherent $X_K$-modules $M(X_K)$ is equivalent to the factor category $M(X_R)/B$, where $B$ is the full subcategory in $M(X_A)$ consisting of the sheaves with support in $X_k \subset X_R$ [1]. Hence we can define the following connecting homomorphisms [7]:

$$\partial: K^\cdot_{+1}(X_K) \to K^\cdot(B) = K^\cdot(X_k).$$

The composition

$$s_\pi: K^\cdot(X_K) \to K^\cdot_{+1}(X_K) \xrightarrow{\partial} = K^\cdot(X_k).$$
where the first homomorphism is the multiplication by the
inverse image of the prime element $\pi$ in the map
$K_1(K) \to K_1(X_k)$ is called the specialization homomorphism.

2. The category of finitely generated $D_K$-modules $D_K$-mod is
equivalent to the factor category $D_R$-mod//C, where C is the full
subcategory in $D_R$-mod, consisting of all torsion $D_R$-modules.
Hence we can define the following connecting homomorphism
[7]:

$$\partial: K_{*+1} + (D_k) \to K_*(C) = K_*(D_k).$$

The composition

$$s_{\pi}: K_*(D_K) \to K_{*+1}(D_K) \to K_*(D_k),$$

where the first homomorphism is the multiplication by the
prime element $\pi$ is also called the specialization homomorphism.

3. The exact sequence of complexes

$$0 \to \bigcup_{x \in X_{k,*+1}} K_*F(x) \to \bigcup_{x \in X_{k,*}} K_*F(x) \to \bigcup_{x \in X_k} K_*F(x) \to 0$$

induces the following connecting homomorphism

$$\partial: H^i(X_K, K_{*+1}) \to H^i(X_k, K_*).$$

The composition

$$s_{\pi}: H^i(X_K, K_*) \to H^i(X_K, K_{*+1}) \xrightarrow{\partial} H^i(X_k, K_*),$$
where the first homomorphism is the multiplication by $\pi \in H^0(X, K_1)$ is also called the specialization homomorphism.

Let $u \in H^i(X, K_1)$; for any field extension $L/F$ by $u_L \in H^i(X_L, K_1)$ we denote the image of $u$ under the homomorphism $H^i(X, K_1) \to H^i(X_L, K_1)$.

**Lemma 1.** For any prime element $\pi$ of the ring $R$ the equality $s_\pi(u_K) = u_k$ holds.

**Proof.** By the product formula $s_\pi(u_K) = \partial(u_{K^*}\pi) = u_k \cdot \partial(\pi) = u_k$

since $\partial(\pi) = 1 \in H^0(X, K_0)$.

**Example.** Let $C$ be an irreducible curve over the field $F$, $c \in C$ be a nonsingular point, $R = 0_{C,c}$ be the local ring of the point $c$. In this case $k = F(c)$, $K = F(C)$ and we have the following

**Corollary.** For any nonsingular rational point $c \in C$, prime element $\pi \in 0_{C,c}$ and $u \in H^i(X, K_1)$ the equality $s_\pi(u_{F(c)}) = u$ holds, i.e. the result of the specialization in this case does not depend on the choice of $c$ and $\pi$.

The category $M(X)$ has the following filtration:

$M(X)_0 \subset M(X)_1 \subset \ldots \subset M(X)_d = M(X)$ where $d = \dim X$ and $M(X)_i$ is the full subcategory in $M(X)$ consisting of all sheaves $G$ such that $\dim \text{supp } G \leq i$. Since $K_*(M(X)_i/M(X)_{i-1}) = \bigcup_{x \in X_i} K_* F(x)$ [7] the inclusion $M(X)_0 \subset M(X)$ induces the homomorphism $t: H^0(X, K_1) \to K^*(X)$.

**Lemma 2.** For any discrete valuation ring $R$ with the fraction field $K$ and the residue field $k$ for any prime element $\pi \in R$ the following diagram
$$H_0(X_K, K^*) \xrightarrow{t} K^*(X_K) \quad \downarrow s_\pi \quad \downarrow s_\pi$$

$$H_0(X_k, K^*) \xrightarrow{t} K^*(X_k)$$

is commutative.

Proof. It is clearly sufficient to prove the commutativity of the following diagram:

$$H_0(X_K, K^*_{+1}) \xrightarrow{t} K^*_{+1}(X_K) \quad \downarrow \partial' \quad \downarrow \partial''$$

$$H_0(X_k, K^*) \xrightarrow{t} K^*(X_k).$$

Since $M(X_R)_0 = M(X_k)_0$ the functor $M(X_R)_1 \rightarrow M(X_R) \rightarrow M(X_K)$ induces the functor $M(X_R)_1/M(X_R)_0 \rightarrow M(X_K)_0 \rightarrow M(X_K)$. Therefore we have the following commutative diagram

$$M(X_R)_0 \rightarrow M(X_R)_1 \rightarrow M(X_R)_1/M(X_R)_0$$

$$\quad \uparrow \quad \uparrow \quad \downarrow$$

$$M(X_R)_0 \downarrow \quad M(X_K)_0 \downarrow$$

$$M(X_k) \rightarrow M(X_R) \rightarrow M(X_K),$$

which induces the commutative diagram

$$K^*_{+1}(M(X_R)_1/M(X_R)_0) \xrightarrow{\partial} K^*(M(X_R)_0) \quad \uparrow$$

$$\downarrow$$

$$K^*_{+1}(M(X_K)_0) \quad K^*(M(X_k)_0) \quad \downarrow$$

$$K^*_{+1}(X_K) \quad \partial'' \quad \rightarrow \quad K^*(X_k).$$
By definition of $\partial'$ the following diagram

\[
\begin{array}{ccc}
K_{*+1}(M(X_R)_1/M(X_R)_0) & \xrightarrow{\partial} & K_{*}(M(X_R)_0) \\
\downarrow & & \parallel \\
K_{*+1}(M(X_K)_0) & \xrightarrow{\partial} & K_{*}(M(X_K)_0) \\
\downarrow & & \downarrow \\
H_0(X_K,,K_{*+1}) & \xrightarrow{\partial'} & H_0(X_K,K_{*})
\end{array}
\]

is commutative. Comparing the two last diagrams we get the result we need.

2. The definition of $p_n: H_0(X(D),K_n) \to K_nD$

Let $X = X(D)$ be a Severi-Brauer variety over a field $F$, associated to the central simple $F$-algebra $D$ of dimension $n^2$, $J$ be the canonical locally free $O_X$-module of rank $n$, and $D = \text{End}_X(J)$ [7,9].

For any commutative $F$-algebra $B$ consider the full subcategory $M'(X_B)$ in $M(X_B)$ consisting of $X_B$-modules $G$ such that $R^i f_*(J \otimes_X G) = 0$ for any $i > 0$, where $f: X_B \to \text{Spec} B$ is the structural morphism. By this theorem of Quillen [7] the inclusion $M'(X_B)$ in $M(X_B)$ induces an isomorphism $K_*(M'(X_B)) \to K_*(X_B)$.

It is clear that for any $G \in M(X_B)$ $B$-module $f_*(J \otimes_X G)$ has a structure of the left $D_B$-module. The exact functor

\[ j_B: M'(X_B) \to D_B - \text{mod}, G \mapsto f_*(J \otimes_X G) \]

induces the homomorphism $K_*(M'(X_B)) \to K_*(D_B)$. We define $p_B$ as a composition.
The Group of $K_1$-Zero-Cycles on Severi-Brauer Varieties

\[ p_B: H_0(X_B, K^*) \xrightarrow{i} K^*(X_B) = K^*(M'(X_B)) \to K^*(D_B) \]

Let $R$ be a discrete valuation ring with the fraction field $K$ and the residue field $k$, and $\pi \in R$ be any prime element. The following statement shows that the homomorphism $p_K$ and $p_k$ are compatible with the specialization.

**Proposition 1.** The diagram

\[
\begin{array}{ccc}
H_0(X_K, K^*) & \xrightarrow{p_K} & K^*(D_K) \\
\downarrow{s_\pi} & & \downarrow{s_\pi} \\
H_0(X_k, K^*) & \xrightarrow{p_k} & K^*(D_k)
\end{array}
\]

is commutative.

**Proof.** By Lemma 2 it is sufficient to prove that the following diagram is commutative

\[
\begin{array}{ccc}
K_{\ast+1}(X_K K) & \to & K_{\ast+1}(D_K) \\
\downarrow{\partial} & & \downarrow{\partial} \\
K^*(X_k) & \to & K^*(D_k).
\end{array}
\]

But this follows from the commutative diagram of functors

\[
\begin{array}{ccc}
M'(X_k) \to & M'(X_R) \to & M'(X_K) \\
\downarrow{j_k} & \downarrow{j_R} & \downarrow{j_K} \\
D_k - \text{mod} \to & D_R - \text{mod} \to & D_k - \text{mod}.
\end{array}
\]

Let now $B = F, x \in X$ be any closed point. We want to compute the following composition

\[ r_x: K_*(F(x)) \to H_0(X, K^*) \xrightarrow{p} K^*(D), \]
where $p = p_F$. Consider the diagram of functors
\[
\begin{array}{ccc}
F(x)\text{-mod} & \to & D_F(x)\text{-mod} \\
\downarrow i_* & & \downarrow \\
M'(X) & \overset{j}{\to} & D\text{-mod},
\end{array}
\]

where $i: \text{Spec} F(x) \to X$ is the closed immersion, right functor is induced by the inclusion $D \subset D_F(x)$ and the top arrow sends $F(x)$-module $M$ to $D_{F(x)}$-module $j(x) \otimes_{F(x)} M$.

Since $\dim_{F(x)} j(x) = n$, $j(x)$ is a simple $D_{F(x)}$-module and therefore the top arrow is the equivalence of categories. The commutativity of the diagram following from the natural isomorphism $j \otimes_x (i_* M) = i_* \left( j(x) \otimes_{F(x)} M \right)$ for any $F(x)$-module $M$ shows that $r_x$ is induced by the functor
\[
F(x)\text{-mod} \to D\text{-mod}; M \mapsto j(x) \otimes_{F(x)} M
\]

and therefore can be decomposed as follows:
\[
r_x: K^* F(x) \to K^* D_{F(x)} \to K^*(D),
\]

where the first map is an isomorphism induced by the equivalence of categories and the second map is the homomorphism of transfer.

Let now $D$ be a skew field and $x$ be a point of degree $n$. We embed $F(x)$ in $D$ as a maximal subfield. Since $F(x)$-modules $j(x)$ and $D$ are isomorphic, $j(x) \otimes_{F(x)} M = D \otimes_{F(x)} M$ for any $F(x)$-module $M$ and therefore the homomorphism $r_x: K^* F(x) \to K^*(D)$ is induced by the inclusion of $F(x)$ in $D$.

**Lemma 3.** If $D$ is split then $p: H_0(X,K^*) \to K^*(D)$ is an isomorphism.
Proof. In this case $X = PF^{n-1}$ is the projective space. Let $x \in X$ be any rational point. In the commutative diagram

\[
\begin{array}{ccc}
K \cdot F(x) & \xrightarrow{\sim} & K \left( D_F(x) \right) \\
\downarrow & & \downarrow \\
H_0(X, K) & \xrightarrow{p} & K \cdot D
\end{array}
\]

the vertical maps are isomorphisms since $X$ is a projective space [8], $F(x) = F$ and therefore $p$ is an isomorphism too.

Now we formulate the main result of the present paper.

Theorem. Let $X$ be a Severi-Brauer variety corresponding to the central simple algebra $D$. Then the homomorphism $p_1: H_0(X, K_1) \to K_1(D)$ is an isomorphism.

The rest of the paper is devoted to the proof of this theorem.

3. The map $q: S(D) \to H_0(X(D), K_1)$

The idea is to construct the inverse map to $p = p_1$. In this section we build the "first approach" of this inverse map.

Let $R$ be a commutative ring, $B$ be an Azumaya algebra over $R$ of rank $n^2$, and $X = X(B)$ be a Severi-Brauer scheme associated to $B$. For any commutative $R$-algebra $S$ the set $X(S)$ of $S$-points of $X$ coincides with the set of direct summands of the rank $n$ of $S$-module $B \otimes_R S$ which are right ideals [9].

Let $A \subset B$ be a commutative $R$-subalgebra in $B$. Considering $B$ as an $A$-module with respect to the right multiplication define the following homomorphism

\[ f: B \otimes_R A \to \text{End}_A(B); f(x \otimes a)(b) = xba. \]

Suppose that
1. $A$ is the direct summand of the $A$–module $B$.
2. $f$ is an isomorphism.

Then $A$–module $B = \text{Hom}_A(A,B)$ is the direct summand of the projective $A$–module $\text{End}_A(B) = B \otimes R A$ and therefore is projective. Since $f$ is an isomorphism, rank $_AB = n$. Hence $\text{Hom}_A(B,A)$ is the right ideal of rank $n$ and the direct summand in $\text{End}_A(B) = B \otimes R A$ and therefore defines the element in the set of points $X(A)$, i.e. the morphism $\text{Spec}A \to X$.

Note that this construction is functional: for any $R$–algebra $S$ the subalgebra $A \otimes R S$ in $B \otimes R S$ satisfies the conditions 1 and 2 and the corresponding morphism $\text{Spec}(A \otimes_R S) \to X_S$ is the base change in the morphism $\text{Spec}A \to X$.

Let $D$ be a central skewfield of dimension $n^2$ over a field $F$, $L \subset D$ be a maximal subfield. Then the subalgebra $A = L$ satisfies 1 and 2 [6] and therefore defines the morphism $\text{Spec}L \to X = X(D)$. Denote by $x \in X$ the image of the unique point in $\text{Spec}L$. Since the field $F(x)$ splits $D$, we have $[F(x):F] \geq n$. On the other hand, our morphism induces the embedding $F(x)$ in $L$. Therefore this embedding is an isomorphism. We'll denote the point $x$ by $[L]$. So $[L]$ is the closed point of degree $n$ with the residue field isomorphic to $L$.

Let $u \in D$; the ring $F[u]$ generated by $u$ over $F$ is a subfield in $D$. We define the set $S(D)$ of all elements $u \in D^*$ such that $F[u]$ is the maximal subfield in $D$. Since there exists a separable over $F$ maximal subfield [6] and this subfield is generated by one element, the set $S(D)$ is not empty. Note also that $u \in S(D)$ if and only if Cayley-Hamilton polynomial of $u$ [4] is irreducible.

Define the following map:

$$q : S(D) \to H_0(X,K_1)$$

by the formula $q(u) = u[L]$ where $L = F[u]$ (we identify $L$ and the residue field of the point $[L]$).
Lemma 4. For any \( u \in S(D) \) the following equality holds:

\[
p(q(u)) = u \mod [D^*, D^*] \in K_1D = D^*/[D^*, D^*].
\]

Proof. Let \( x = [L] \); the results of Section 2 imply that the composition \( L^* = F(x)^* \mapsto H_0(X, K_1) \mapsto K_1(D) \) is induced by the embedding \( L \) to \( D \). Therefore \( p(q(u)) = p(ux) = u \mod [D^*, D^*] \) in \( K_1D \).

Now consider the behavior of \( q \) under the specialization. We take an affine line \( A^1 = \text{Spec} F[T] \), rational point \( T = t \in F \) with a local ring \( R = F(T)_{(T-t)} \) and the prime element \( \pi = T - t \). It is clear that \( R/\pi R = F \) and \( F(T) \) is the fraction field of \( R \). Consider the specialization map \( s_\pi \) associated with the discrete valuation ring \( R \).

Proposition 2. Let \( S_t \) be the set of all polynomials \( u(T) \in D[T] \) such that \( u(t) \in S(D) \). Then \( S_t \subset S(D) \) and we have commutative diagram

\[
\begin{array}{ccc}
S_t & \longrightarrow & S(D) \\
\downarrow q_{F(T)} & & \downarrow q \\
H_0(X(D_{F(T)}), K_1) & \xrightarrow{s_\pi} & H_0(X(D), K_1)
\end{array}
\]

where the above homomorphism is the "value in the point \( T = t \)."

Proof. Let \( u(T) \in S_t, P(T, X) \in F[T, X] \) be Cayley-Hamilton polynomial of \( u(T) \) as an element of Azumaya algebra \( D[T] \) over \( F[T] \). Since the polynomial \( P(t, X) \) is irreducible, \( P(T, X) \) is also irreducible and \( u(T) \in S(D(T)) \).

The homomorphism \( s_\pi \) coincides with the composition
\[ H_0 \left( X \left( D_{F(T)}, K_1 \right) \right) \xrightarrow{\pi} H_0 \left( X \left( D_{F(T)}, K_2 \right) \right) \xrightarrow{\partial} H_0 \left( X(D), K_1 \right). \]

Hence it is sufficient to prove the equality \( \partial([u(T), T-t][E]) = u(t)[L] \) where \( E = F(T)[u(T)], L = F[u(t)] \).

Denote the ring \( R[u(T)] \) by \( A \). It is clear that \( A \) is a discrete valuation ring with the prime element \( \pi \), fraction field \( E \) and residue field \( L \). We consider \( A \) as a commutative subalgebra in Azumaya \( R \)-algebra \( B = D \otimes_F R \) and show that the canonical homomorphism \( f: B \otimes_R A \to \text{End}_A(B) \) is an isomorphism. \( L \) is the maximal subfield in \( D \), hence \( f \) is an isomorphism modulo maximal ideal of \( R \) and by Lemma of Nakayama \( f \) is surjective. Since \( E \) is the maximal subfield in \( D(T) \), the localization \( S^{-1}f \) with respect to the multiplicative set \( S \) of nonzero elements in \( R \) is an isomorphism. Therefore \( f \) is injective and hence is an isomorphism.

Since \( A/\pi A \cong B/\pi B \) \( A \)-module \( B/A \) is torsionfree and therefore \( B/A \) is free \( A \)-module and \( A \) the direct summand in \( B \).

So we have shown that algebra \( B \) and commutative subalgebra \( A \) satisfy the conditions 1 and 2 and define the morphism \( \text{Spec} A \to X(B) \). The functional property gives us the commutative diagram

\[
\begin{array}{ccc}
\text{Spec}L = \text{Spec} A/\pi A & \to & X(B/\pi B) = X(D) \\
\downarrow & & \downarrow \\
\text{Spec}A & \to & X(B) \\
\uparrow & & \uparrow \\
\text{Spec}E = \text{Spec}(S^{-1}A) & \to & X(S^{-1}B) = X(D(T))
\end{array}
\]

which induces the following commutative diagram
\[ K_2 E = H_0(\text{Spec} E, K_2) \rightarrow H_0\left( \mathbb{X}(D_{F(T)}), K_2 \right) \]
\[ \downarrow \vartheta \quad \downarrow \vartheta' \quad \downarrow \vartheta'' \]
\[ K_1 L = H_0(\text{Spec} L, K_1) \rightarrow H_0(\mathbb{X}(D), K_1) \]

where \( \vartheta \) is the tame symbol associated to a discrete valuation ring \( A \). In particular \( \vartheta(\{ u(T), T-t \}) = u(t) \).

Consider another example of the specialization.

**Proposition 3.** Let \( K \subset D \) be a maximal subfield, \( u(T) \in K[T], u(t) \neq 0 \). Then \( s_n(u(T)[K(T)]) = u(t)[K] \).

**Proof.** The functional property gives us the commutative diagram

\[
\begin{array}{ccc}
\text{Spec} K(T) & \rightarrow & X_{F(T)} \\
\downarrow & & \downarrow j \\
\text{Spec} K & \rightarrow & X
\end{array}
\]

Denote \([K]\) by \( x \in X\) and \([K(T)]\) by \( y \in X_{F(T)}\). The projection \( j \) is decomposed into the composition \( X_{F(T)} \xrightarrow{r} X \times A^1 \rightarrow X \) and the closure of the point \( r(y) \) in \( X \times A^1 \) equals \( x \times A^1 \). Since \( u(T) \in K[T] = F[x \times A^1] \) is a regular functor on \( x \times A^1 \), the specification map sends the element \( u(T)y \) at first by the multiplication on \( \pi = T-t \) in \( (u(T), T-t)y \) and then by \( \vartheta \) to the element \( u(t)x \).

4. The construction of the homomorphism \( q : K_1(D) \rightarrow H_0(\mathbb{X}(D), K_1) \)

In this section we show how to extend the map \( q \) constructed in Section 3 from the "dense" subset \( S(D) \) to the whole group \( D^* \). This extension modulo the commutant appears to be the inverse map to \( p = p_1 \).
We begin with the following abstract situation. Let $G$ be any group; a subset $S \subset G$ is called dense in $G$ if for any elements $g_1, g_2, \ldots, g_n \in G$ the intersection $\bigcap Sg_i$ is not empty.

Lemma 5. Let $S$ be a dense subset in group $G$ such that $S = S^{-1}$ and $q: S \to B$ be a map to abelian group $B$. Suppose that

1. $q(g^{-1}) = -q(g)$ for any $g \in G$.

2. $q(g_1g_2) = q(g_1) + q(g_2)$ for all $g_1, g_2 \in S$ such that $g_1g_2 \in S$.

Then there exists the unique homomorphism $q': G \to B$ extending the map $q$.

Proof. Let $g \in G$; since $Sg \cap S1 \neq \emptyset$, we have: $sg = t \in S$ for some $s \in S; g = s^{-1}t$. If $q'$ extends $q$ then $q'(g) = -q(s) + q(t)$ which proves the uniqueness.

Now we prove the existence of the extension. Let $g \in G$; as before we find $s, t \in S$ such that $g = s^{-1}t$. We define $q'$ by the formula $q'(g) = -q(s) + q(t)$. To prove that $q'$ is well defined, take $g = s_1^{-1}t_1$ where $s_1, t_1 \in S$. Choose $s_2 \in Ss \cap Ss_1 \cap Sg^{-1} \cap S1$ then $g = s_2^{-1}t_2, t_2 \in S$ and $s_2s_1^{-1} = t_2t_1^{-1} \in S$, $s_2s_1^{-1} = t_2t_1^{-1} \in S$. Therefore

$$-q(s) + q(s_2) = q(s_2s_1^{-1}) = q(t_2t_1^{-1}) = q(t_2) - q(t_1),$$

$$-q(s_1) + q(s_2) = q(s_2s_1^{-1}) = q(t_2t_1^{-1}) = q(t_2) - q(t_1),$$

hence $-q(s) + q(t) = -q(s_1) + q(t_1)$ which proves that $q'$ is well defined.

If $g \in S$ and $g = s^{-1}t$ for $s, t \in S$, then $q'(g) = -q(s) + q(t) = q(s^{-1}t) = q(g)$, i.e. $q'$ is the extension of $q$.

Finally we have to show that $q'(gh) = q'(g) + q'(h)$ for any $g, h \in G$. Suppose at first that $g \in S$. Choose $s \in Sg \cap Sh^{-1} \cap S1$ then
\[ h = s^{-1}t, t \in S \text{ and } gs^{-1} \in S. \text{ We have: } q'(g) + q'(h) = q(g) - q(s) + q(t) = q(gs^{-1}) + q(t) = q'(gh) \text{ since } gh = (sg^{-1})^{-1}t. \text{ Now consider the general case. Choose } t \in Sg \cap Sh^{-1} \cap S1 \text{ i.e., } s^{-1}t = g, s \in S \text{ and } t \in S. \text{ Using the first case we have: } q'(g) + q'(h) = -q(s) + q(t) + q'(h) = -q(s) + q'(th) = -q(s) + q(th) = q'(gh) \text{ since } gh = s^{-1}th. \]

Remark. It follows from the proof that \( G \) is generated by any dense subset.

Let \( D \) be a central skewfield of dimension \( n^2 \) over a field \( F, G = D^*, S = S(D) \subset G. \)

Lemma 6. The set \( S \) satisfies the conditions of Lemma 5, i.e. \( S^{-1} = S \) and \( S \) is dense in \( G \).

Proof. Since \( F[u^{-1}] = F[u], S^{-1} = S. \) If \( F \) is a finite field, the skewfield \( D \) is trivial [6] and therefore \( S = G \) is dense in \( G \).

Suppose now that \( F \) is an infinite field. Note that the set \( S \) is open in Zariski topology of affine space \( D = A \dim D. \) Indeed, \( u \in S \) if and only if the elements \( 1, u, u^2, \ldots, u^{n-1} \in D \) are linearly independent over \( F \) if the rank of the matrix of coefficients of these elements in some basis of \( D \) is lesser than \( n \), i.e. the set \( D-S \) is closed in \( D \) and \( S \) is open. Therefore, for any \( g_1, g_2, \ldots, g_n \) in \( G \) the sets \( Sg_i \) are open and nonempty and since the field \( F \) is infinite, the intersection of these sets is not empty, i.e. \( S \) is dense in \( G \).

Now consider the abelian group \( B = H_0(X(D), K_1) \) and the map \( q:S \to B \) defined in Section 3. We prove that \( q \) satisfies the conditions of Lemma 5. Let \( u \in S, L = F[u] \); since \( F[u^{-1}] = L, q(u^{-1}) = (u^{-1})(L) = -q(L) = -q(u). \)

Finally we have to show that \( q(uv) = q(u) + q(v) \) for \( u, v \in S \) such that \( uv \in S \). Denote the polynomial \( vT + 1 - T \) by \( v(T). \) Since \( v(1) = v \in S(D), \) it is clear that \( uv(T) \in S\left(D_{F(T)}\right). \) Consider the element \( w = q(u) + q(v(T)) - q(uv(T)) \in H_0\left(X_{F(T)}, K_1\right). \) By Lemma 4 \( p(w) = uv(T)(uv(T))^{-1} = 1 \in K_1(D(T)). \)
Lemma 7. Let \( u \in \ker\left( H_0\left( X_{F(T)}, K_1 \right) \rightarrow P \rightarrow K_1D_{F(T)} \right) \). Then the image of the specialization \( s_\pi(u) \in H_0(X, K_1) \) in the rational point \( T = t \in F \) does not depend on the choice of \( t \) and \( \pi \).

Proof. Let \( L/F \) be any splitting field of \( D \). From the commutative diagram

\[
\begin{array}{ccc}
H_0\left( X_{F(T)}, K_1 \right) & \xrightarrow{P} & K_1D_{F(T)} \\
\downarrow i & & \downarrow \\
H_0\left( X_{L(T)}, K_1 \right) & \rightarrow & K_1D_{L(T)}
\end{array}
\]

and Lemma 3 we get that \( u \in \ker i \). The exact sequence of complexes

\[
0 \rightarrow \bigcup_{y \in A^1} \bigcup_{x \in X_{F(y)}^{*^{-1}}} K_s F(x) \rightarrow \bigcup_{x \in (X \times A^1)^*} K_s F(x) \rightarrow \bigcup_{x \in X_{F(T)}^{*}} K_s F(x) \rightarrow 0
\]

and isomorphism \( H_1(X \times A^1, K_2) = H_0(X, K_1) \) [8] give us the commutative diagram with the exact top row

\[
\begin{array}{ccc}
H_0(X, K_1) & \xrightarrow{k} & H_0\left( X_{F(T)}, K_1 \right) \\
\downarrow i & & \downarrow i \\
& \bigcup_{y \in A^1} H_0\left( X_{F(y)}, K_0 \right) & & \bigcup_{y \in A^1} H_0\left( X_{L(y)}, K_0 \right)
\end{array}
\]

The homomorphism \( j \) is injective by the theorem of Panin [5]. Therefore \( u \in \text{im}(k) \) and we can apply the Corollary to Lemma 1.

By Lemma 7 and Propositions 2 and 3 we have:

\[
q(u) + q(v) - q(uv) = s_{T-1}(w) = s_T(w) = q(u) - q(u) = 0
\]
So we can apply Lemma 5 to construct the extension of $q$:

$$q' : D^* \to H_0(X(D), K_1)$$

which clearly factors through the homomorphism

$$K_1(D) \to H_0(X(D), K_1)$$

that we'll denote by $q$.

Since the set $S(D)$ generates $D^*$, the composition poq is identified by Lemma 4. In the rest of this section we prove that $q$ commutes with the specialization.

Lemma 8. For any $t \in F$ the group $D_{F(T)^*}$ is generated by the element $T - t$ and set $S_t$.

Proof. It is clear that $D_{F(T)^*}$ is generated by $T - t$ and the set of polynomials $u(T) \in D[T]$ such that $u(t) \neq 0$. Since $S(D)$ is a dense subset in $D^*$, we can find $v \in S(D)$ such that $u(t)v \in S(D)$. Then $v, u(T)v \in S_t$ and $u(T) = (u(T)v)v^{-1}$.

Proposition 4. For any $t \in F$ the diagram

$$
\begin{array}{ccc}
K_1D_{F(T)} & \xrightarrow{s_{T-t}} & K_1D \\
\downarrow q_T & & \downarrow q \\
H_0\left(X_{F(T)}, K_1\right) & \xrightarrow{s_{T-t}} & H_0(X, K_1)
\end{array}
$$

is commutative.

Proof. By Lemma 8 the group $D_{F(T)^*}$ is generated by $T - t$ and the set $S_t$. The commutativity for the elements of the set $S_t$ was proved in Proposition 2. Instead of element $T - t$ it is sufficient to consider $(T - t)u$, where $u$ is any element in $S(D)$. Let $L = F[u]$; then $F(T) [T - t]u] = L(T)$ and
\[(s_{T^{-1}} \circ g_T)((T-t)u)) = S_{T^{-1}}(((T-t)u[L(T)]) = \partial(((T-t)u,T-t)[L(T)])
= \partial([-u,T-t][L(T)]) = (-u)[L],\]
\[(g \circ s_{T^{-1}})((T-t)u) = g(\partial(((T-t)u,T-t))) = g(-u) = (-u)[L].\]

5. Proof of the Theorem

We have only to show that the composition \( q \circ p \) is identity. Let \( x \in X \), and \( u \in F(x)^* \). Consider the point \( \overline{x} \in X_{F(T)} \) over \( x \) and generic point of \( \text{Spec}F(T) \) and an element \( \overline{u} = uT + 1 - T \in F(T)(\overline{x}) = F(x)(T) \). Denote by \( w \) the element \( q(p(\overline{ux})) - \overline{ux} \in H_0(X_{F(T)}, K_1) \). Since \( p(w) = 0 \), Lemma 7 all the specializations of \( w \) in rational points coincide; in particular \( s_{T^{-1}}(w) = s_T(w) \). By Propositions 1 and 4 the homomorphisms \( p \) and \( q \) commute with the specialization and we have:
\[ s_{T^{-1}}(w) = q(p(s_{T^{-1}}(\overline{ux}))) - s_{T^{-1}}(\overline{ux}) = q(p(\overline{ux})) - \overline{ux} \text{ since } s_{T^{-1}}(\overline{ux}) = \overline{ux} \]
and \( s_T(w) = q(p(s_T(\overline{ux}))) - s_T(\overline{ux}) = 0 \text{ since } s_T(\overline{ux}) = 0 \). Therefore,
\[ q(p(\overline{ux})) = \overline{ux}, \text{i.e. } q \circ p = id. \]

So we have proved the Theorem in the case when \( D \) is a skewfield. Now let \( A \) be any central simple \( F \)-algebra, \( A = M_m(D) \) where \( D \) is a skewfield. Using the results of [5] one can find a closed subvariety \( Z \subset X(A) \) such that \( Z \cong X(D) \) and a vector bundle \( X(A) - Z \rightarrow X(A') \) where \( A' = M_{m-1}(D) \). Therefore, \( H_0(X(A)-Z,K_1) = 0 \) and the direct image
\[ H_0(X(D),K_1) = H_0(Z,K_1) \rightarrow H_0(X(A),K_1) \]
is a surjective map. The Theorem follows from the commutative diagram.
The Group of $K_1$-Zero-Cycles on Severi-Brauer Varieties

\[ H_0(X(D), K_1) \rightarrow K_1(D) \]
\[ \downarrow_{i*} \quad \downarrow \]
\[ H_0(X(A), K_1) \rightarrow K_1(A). \]

References


Received November 26, 1991