1. Let $N$ be the normal subgroup in $G * H$ generated by $G \subset G * H$. Prove that $(G * H)/N \simeq H$.

2. Let $D_\infty = \mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z})$ with respect to the (unique) isomorphism $\mathbb{Z}/2 \simeq \text{Aut}(\mathbb{Z})$. Prove that $D_\infty \simeq (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$.

3. Show that there exists a surjective homomorphism $(\mathbb{Z}/n\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) \to S_n$.

4. Prove that the group $SL_2(\mathbb{Z})$ is generated by the two matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

5. Let $H$ and $K$ be two subgroups in $G$. Assume that $G$ acts on a set $X$ and there are two subsets $A \subset X$, $B \subset X$ and an element $x \in X \setminus (A \cup B)$ such that $h(A \cup \{x\}) \subset B$ for every $h \in H$, $h \neq e$ and $k(B \cup \{x\}) \subset A$ for every $k \in K$, $k \neq e$. Prove that the natural homomorphism $H * K \to G$ is injective.

6. Let $G$ and $H$ be two non-trivial groups. Show that $G * H$ is an infinite group with trivial center.

7. Let $X$ be a subset in a group $G$. Prove that $\langle \langle X \rangle \rangle = \langle Y \rangle$, where $Y$ is the union of $gXg^{-1}$ for all $g \in G$ ($\langle \langle X \rangle \rangle$ denotes the normal subgroup generated by $X$).

8. Let $G$ be the group defined by generators $a$, $b$ and relations $w^3 = e$ for all words $w$ in $a$ and $b$. Show that $G$ is finite and find $|G|$.

9. (a) Prove that if $\mathbb{Z}^n \simeq \mathbb{Z}^m$, then $n = m$.

(b) Prove that if $F(X) \simeq F(Y)$ for finite sets $X$ and $Y$, then $|X| = |Y|$.

10. Show that the group $\mathbb{Q}/\mathbb{Z}$ is not finitely generated.