1. Let $\sigma = (123 \cdots n) \in S_n$. Show that the conjugacy class of $\sigma$ has $(n - 1)!$ elements. Show that the centralizer of $\sigma$ is the cyclic subgroup generated by $\sigma$.

2. Prove the following Useful Counting Result. Let $H$ be a subgroup of a finite group $G$. Suppose that $|G|$ does not divide $[G : H]$!. Then $G$ contains a proper normal subgroup $N$ such that $N$ is a subgroup of $H$. In particular, $G$ is not simple.

3. Prove that all groups of order $2p^n$ and $4p^n$ ($p$ a prime) are not simple.

4. a) Let $H \subseteq G$ be a subgroup. Prove that if $H$ is contained in the center of $G$ and the factor group $G/H$ is cyclic, then $G$ is abelian.
   (b) Prove that any group of order $p^2$ ($p$ a prime) is abelian.

5. Let $G$ be a nonabelian group of order $p^3$ ($p$ prime). Prove that the center $Z(G)$ of $G$ coincides with the commutator subgroup $[G, G]$.

6. Let $G$ be a semidirect product of a cyclic normal subgroup $N$ of order $n$ and an abelian group $K$. Show that if $|K|$ is relatively prime to $\varphi(n)$ ($\varphi$ is the Euler function), then $G$ is abelian.

7. Determine center of the dihedral group $D_n$.

8. Find all normal subgroups of $D_n$.

9. Show that the exact sequence
   
   $$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

   does not split.

10. For every two nonzero integers $n$ and $m$ construct an exact sequence

   $$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/nm\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0.$$ 

For which $n$ and $m$ does the sequence split?