Homework 3 (Due: Fr, 4/27)

Problem 1: Let $X$ be a topological space. A presheaf $S$ of abelian groups on $X$ consists of the following data:

(i) an abelian group $S(U)$ for each open set $U \subseteq X$,

(ii) a group homomorphism $r^U_V : S(U) \to S(V)$ for each pair $U, V \subseteq X$ of open sets with $V \subseteq U$. Here we require that $r^U_U = \text{id}_{S(U)}$ for $U \subseteq X$ open, and that

$$r^V_W \circ r^U_V = r^U_W$$

whenever $U, V, W \subseteq X$ are open and $W \subseteq V \subseteq U$.

The homomorphism $r^U_V$ is called the restriction homomorphism from $U$ to $V$, and one usually writes $f|_V$ for $r^U_V(f)$ if $f \in S(U)$.

The presheaf $S$ is called a sheaf if the following additional conditions are satisfied: Suppose $U \subseteq X$ is any open set that it is written as a union $U = \bigcup_{i \in I} U_i$ of open sets $U_i \subseteq X$. Then we require:

(iii) If $f, g \in S(U)$ and $f|_{U_i} = g|_{U_i}$ for all $i \in I$, then $f = g$.

(iv) If $f_i \in S(U_i)$ and $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists $f \in S(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

Note that by (iii) the element whose existence is required in (iv) is uniquely determined. Applying (iii) to $U = \emptyset$ and $I = \emptyset$, we conclude that if $S$ is a sheaf, then $S(\emptyset)$ consists of precisely one element, and so $S(\emptyset) = \{0\}$.

One defines presheaves and sheaves of rings, vector spaces, etc., similarly.

a) Let $X \subseteq \mathbb{C}$ be open, and for $U \subseteq X$ let $\mathcal{O}(U)$ be the ring of holomorphic functions on $U$. Show that this together with the homomorphisms given by restrictions of functions defines a sheaf on $X$ (the sheaf of holomorphic functions on $X$, usually denoted by $\mathcal{O}$ with $X$ understood).

b) Let $X$ be a topological space, and $G$ be an abelian group. For an open set $U \subseteq X$ let $G(U)$ be the set of locally constant functions on $U$ with values in $G$. Show that this together with the homomorphisms given by restrictions of functions defines a sheaf on $X$ (by abuse of notation also denoted by $G$).

Let $S$ be a presheaf of abelian groups on $X$, and $x \in X$. Suppose $f \in S(U)$ and $g \in S(V)$, where $U, V \subseteq X$ are open and $x \in U \cap V$. Then we write $f \sim g$ if there exists an open set $W \subseteq X$ with $x \in W \subseteq U \cap V$ and $f|_W = g|_W$.

c) Show that $\sim$ defines an equivalence relation (on the disjoint union of the sets $S(U)$ with $x \in U \subseteq X$).
d) Let $S_x$ be the set of equivalence classes $[f]$ (with respect to $\sim$) of all elements $f \in S(U)$ obtained from open sets $U \subseteq X$ with $x \in U$. Show that $S_x$ carries a natural abelian group structure, and that if $U \subseteq X$ is open and $x \in U$, then the map $r_x : S(U) \to S_x$ sending $f \in S(U)$ to its equivalence class $[f] \in S_x$ is a group homomorphism.

The group $S_x$ is called the **stalk of the sheaf $S$ at $x$**, and the image $r_x(f) \in S_x$ of $f \in S(U)$, where $x \in U$, is called the **germ of $f$ at $x$**.

e) Let $X \subseteq \mathbb{C}$ be open, and $x \in X$.

Show that if $O$ the sheaf of holomorphic functions on $X$, then the stalk $O_x$ can be identified with the set of all power series centered at $x$ with positive radius of convergence.

Show that if $G$ is the sheaf of locally constant functions on $X$ with values in an abelian group $G$, then $G_x$ can be identified with the group $G$.

f) Let $S$ be a sheaf on a topological space $X$, $U \subseteq X$ be open, and $f \in S(U)$.

Show that $f = 0$ if and only if $r_x(f) = 0$ for all germs of $f$ with $x \in U$.

**Problem 2**: Let $X$ be a topological space, $S$ be a sheaf of abelian groups on $X$, and $U = \{U_i : i \in I\}$ be an open cover of $X$. A (Čech) **cochain of degree $n \in \mathbb{N}_0$ (with values in the sheaf $S$)** is a family $\{f_J\}_{J \in I^{n+1}}$, where $J = i_0 \ldots i_n$ runs through all ordered $(n+1)$-tuples of elements in $I$, $f_J \in S(U_{i_0} \cap \cdots \cap U_{i_n})$, and we require the antisymmetry relation

$$f_{i_0 \ldots i_{n+1}} = -f_{i_{n} \ldots i_{0}}$$

for all $(n+1)$-tuples. The set of these cochains forms an abelian group under componentwise addition (i.e., $\{f_J\} + \{h_J\} := \{f_J + h_J\}$) and is denoted by $C^n(U, S)$.

We define a **coboundary operator** $\delta : C^n(U, S) \to C^{n+1}(U, S)$ by sending $f = \{f_J\} \in C^n(U, S)$ to $\delta f \in C^{n+1}(U, S)$ whose components are given by

$$(\delta f)_{i_0 \ldots i_{n+1}} := \sum_{k=0}^{n+1} (-1)^k f_{i_0 \ldots \widehat{i_k} \ldots i_{n+1}}.$$

Here $\widehat{i_k}$ indicates that the index $i_k$ is omitted, and it is understood that the elements on the right-hand side are restricted to $U_{i_0} \cap \cdots \cap U_{i_{n+1}}$. Note that the operator $\delta$ depends on $n$, but this suppressed in the notation.

a) Show that $\delta$ is a well-defined group homomorphism, and that $\delta \circ \delta = 0$.

A cochain $f \in C^n(U, S)$ is called a **cocycle** if $\delta f = 0$, and a **coboundary** if there exists $g \in C^{n-1}(U, S)$ such that $f = \delta g$ (for $n \geq 1$; for $n = 0$ we require $f = 0$ for $f$ to be a coboundary).

b) Show that the coboundaries and cocycles of degree $n$ form subgroups $B^n(U, S)$ and $Z^n(U, S)$ of $C^n(U, S)$, respectively, and that $B^n(U, S) \subseteq Z^n(U, S)$.
c) Show that $\mathcal{S}(X)$ is isomorphic to the group $Z^0(\mathcal{U}, \mathcal{S})$ by finding an explicit isomorphism.

Suppose now that $X$ is an open subset of $\mathbb{C}$, and let $\mathcal{C}^\infty$ be the sheaf of $C^\infty$-smooth functions on $X$ (defined in the obvious way similarly as in Prob. 1 (a)).

d) Show that every cocycle in $C^n(\mathcal{U}, \mathcal{C}^\infty)$, $n \in \mathbb{N}$, is a coboundary. Hint: First consider $n = 1$ to get an idea how to treat the general case.

e) Let $\mathcal{O}$ be the sheaf of holomorphic functions on $X$. Show that every cocycle in $C^n(\mathcal{U}, \mathcal{O})$, $n \in \mathbb{N}$, is a coboundary. Explain how this is related to HW 2, Prob. 4.

**Problem 3:** Let $g$ be a $C^1$-smooth function on $\mathbb{C}$ with compact support. Show that the inhomogeneous Cauchy-Riemann equation

$$\frac{\partial f}{\partial \bar{z}} = g$$

has a $C^1$-smooth solution $f$ on $\mathbb{C}$ with compact support if and only if

$$\int_{\mathbb{C}} z^n g(z) dA(z) = 0 \quad \text{for all } n \in \mathbb{N}_0.$$

**Problem 4:**

a) Let $\Omega \subseteq \mathbb{C}$ be a simply connected region, and $f$ be a meromorphic function on $\Omega$. Suppose that the orders of all zeros and poles of $f$ are even. Show that then there exists a meromorphic function $g$ on $\Omega$ with $f = g^2$. Show that if $h$ is another meromorphic function on $\Omega$ with $f = h^2$, then $h = g$ or $h = -g$.

b) Fix $k \in (0, 1)$ and consider the modular sine $\text{sn}(u) = \text{sn}(u; k)$ for parameter $k$. This is a meromorphic function on $\mathbb{C}$ (cf. 246B, Ex. 16.10, HW8, Prob. 3). Show that $\text{sn}$ has only simple poles, and that if $w_0 \in \mathbb{C}$ and $u_0 \in \mathbb{C}$ satisfy $\text{sn}(u_0) = w_0$, then $\text{sn}$ attains the value $w_0$ at $u_0$ of order 1 if $w_0 \in \mathbb{C} \setminus \{\pm 1, \pm 1/k\}$, and of order 2 if $w_0 \in \{\pm 1, \pm 1/k\}$ (i.e., the function $u \mapsto \text{sn}(u) - w_0$ has a zero at $u_0$ of the corresponding order).

c) Show that there exist unique meromorphic functions $\text{cn}(u) = \text{cn}(u; k)$ and $\text{dn}(u) = \text{dn}(u; k)$ (the modular cosine and the modular delta for parameter $k$) such that $\text{cn}(0) = 1$, $\text{dn}(0) = 1$, and

$$\text{sn}^2(u) + \text{cn}^2(u) = 1 \quad \text{and} \quad \text{dn}^2(u) + k^2 \text{sn}^2(u) = 1$$

for all $u \in \mathbb{C}$.

d) Show that

$$\text{sn}'(u) = \text{cn}(u) \text{dn}(u), \quad \text{cn}'(u) = -\text{sn}(u) \text{dn}(u), \quad \text{dn}'(u) = -k^2 \text{sn}(u) \text{cn}(u)$$

for all $u \in \mathbb{C}$.