Homework 1 (Due: Fr, 4/13)

Problem 1:

a) Suppose $K_n \neq \emptyset$, $n \in \mathbb{N}$, are compact and connected sets in a metric space. Show that if the sequence $\{K_n\}$ is descending, i.e., if $K_{n+1} \subseteq K_n$ for $n \in \mathbb{N}$, then $\bigcap_{n \in \mathbb{N}} K_n$ is a non-empty compact and connected set.

b) Let $U \subseteq \mathbb{C}$ be a region of the form $U = \mathbb{D} \setminus K$, where $K \subseteq \mathbb{D}$ is a compact and connected set with $0 \in K$. Show that then $\Omega := \exp^{-1}(U)$ is a simply connected region. Hint: First show that $\Omega$ is connected.

Problem 2: A ring domain is a region $R \subseteq \mathbb{C}$ that can be written in the form $R = \mathbb{C} \setminus (E \cup F)$, where $E$ and $F$ are non-empty compact and connected subsets of $\mathbb{C}$. The purpose of this problem is to show that every ring domain $R$ is conformally equivalent to one of the regions $\mathbb{C} \setminus \{0\}$ or $A_r := \{z \in \mathbb{C} : r < |z| < 1\}$ with $r \in [0,1)$.

a) Suppose $R$ is a ring domain as above where $\infty \in F$ and $F$ consists of more than one point. Show that then $R$ is conformally equivalent to a region of the form $U = \mathbb{D} \setminus K$, where $K \subseteq \mathbb{D}$ is compact and connected, and $0 \in K$.

b) Show that if $U$ is as in (a), then $\Omega := \exp^{-1}(U)$ is a simply connected region that is invariant under the translation $T: \mathbb{C} \rightarrow \mathbb{C}$, $u \mapsto T(u) = u + 2\pi i$.

c) Show that there exists a conformal map $f$ from $\Omega$ onto the upper half-plane $H = \{v \in \mathbb{C} : \operatorname{Im}(v) > 0\}$ such that for $S = f \circ T \circ f^{-1}$ we have $S(v) = v + a$ with $a \in \mathbb{R} \setminus \{0\}$ or $S(v) = av$ with $a > 0$ for all $v \in \mathbb{C}$.

d) Find a holomorphic covering map $\pi$ of $H$ onto an annulus $A_r$ with $r \in [0,1)$ such that $\pi(v_1) = \pi(v_2)$ for $v_1, v_2 \in H$ if and only if there exists $n \in \mathbb{Z}$ such that $S^n(v_1) = v_2$ (here $S^n$ refers to $n$-th power of $S$ in the group of M"obius transformations).

e) For arbitrary $z \in U$ pick $u \in \exp^{-1}(z)$ and let $F(z) := \pi(f(u))$. Show that $F$ is well-defined and gives a conformal map of $U$ onto $A_r$.

f) Prove the assertion stated in the beginning.

Problem 3:

a) Let $U \subseteq \mathbb{C}$ be a region, $f \in H(U)$, and $\Gamma$ be a piecewise smooth cycle that is null-homologous in $U$. Show that if $|f(z)| \leq 1$ for all $z \in \Gamma^*$, then $|f(z_0)| \leq 1$ for all $z_0 \in U$ with $\operatorname{ind}_\Gamma(z_0) \neq 0$. 

b) Let $\mathcal{F}$ be the set of all $f \in H(D)$ for which
\[
\int_D |f(z)|^2 dA(z) \leq 1.
\]
Show that $\mathcal{F}$ is a normal family.

**Problem 4:** Does there exist a sequence $\{P_n\}$ of polynomials such that $P_n(0) = 1$ for all $n \in \mathbb{N}$, but $P_n(z) \to 0$ as $n \to \infty$ for all $z \neq 0$?