Homework 4 (Due: Fr, 2/10)

Problem 1:

a) Let $k \in \mathbb{N}_0$. Show that
\[
\frac{1}{(1-z)^{k+1}} = \sum_{n=0}^{\infty} \binom{n+k}{n} z^n \quad \text{for } |z| < 1.
\]

b) Find the Laurent series expansion of the function
\[
f(z) = \frac{z^2 - z + 1}{(z-1)(z-2)^2}
\]
converging in
i) $A_1 = \{ z \in \mathbb{C} : 0 < |z-1| < 1 \}$,
ii) $A_2 = \{ z \in \mathbb{C} : 1 < |z-1| \}$.

Problem 2: Show that every holomorphic map $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational function (including constant functions $f \equiv c \in \hat{\mathbb{C}}$).

Problem 3:

a) Consider the function $f(z) := \frac{z}{e^z - 1}$ that is holomorphic in $z \neq 0$ near 0. Show that 0 is a removable singularity of $f$.

b) By a) the function $f$ can be expanded in a power series centered at 0. What is the radius of convergence of this power series?

c) If we write the power series expansion of $f$ near 0 in the form
\[
f(z) = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n,
\]
then the number $B_n$ is called the $n$th Bernoulli number. Show that $B_0 = 1$, and that
\[
B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k \quad \text{for } n \in \mathbb{N}.
\]

d) Show that $B_{2n+1} = 0$ for $n \in \mathbb{N}$.

p.t.o.
e) Show that
\[ \tan z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n} (2^{2n} - 1) B_{2n}}{(2n)!} z^{2n-1} \quad \text{for} \ |z| < \frac{\pi}{2}. \]

**Problem 4:** Fix \( u \in \mathbb{C} \) and consider the function \( f_u(z) := \exp \left( \frac{u}{2} \left( z - \frac{1}{z} \right) \right) \). The function \( f_u \) is holomorphic for \( z \neq 0 \) and has an isolated singularity at 0. Let
\[ f_u(z) = \sum_{n=-\infty}^{\infty} J_n(u) z^n \]
be the Laurent series expansion of \( f_u \) on \( \mathbb{C} \setminus \{0\} \). The coefficient \( J_n(u) \), considered as a function of \( u \), is called the \( n \)th **Bessel function**.

a) Show that
\[ J_n(u) = \frac{1}{\pi} \int_0^{\pi} \cos(u \sin \theta - n\theta) \, d\theta \quad \text{for} \ u \in \mathbb{C}, \ n \in \mathbb{Z}. \]

b) Show that \( J_{-n}(u) = (-1)^n J_n(u) \) for \( u \in \mathbb{C}, \ n \in \mathbb{Z} \).

c) Show that \( J_n \) is an entire function for each \( n \in \mathbb{Z} \).

d) Show that \( J_n(u) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left( \frac{u}{2} \right)^{n+2k} \) for \( u \in \mathbb{C}, \ n \in \mathbb{N}_0 \).