Homework 3 (Due: Fr, 2/3)

Problem 1:

a) Let \( f : \mathbb{D} \to \mathbb{C} \) be a continuous map, and \( \gamma(s) := f(e^{2\pi is}) \) for \( s \in [0, 1] \).
Show that if \( \gamma(s) \neq 0 \) for \( s \in [0, 1] \) and \( \text{ind}_{\mathbb{D}}(0) \neq 0 \), then \( f \) has a zero in \( \mathbb{D} \), i.e., there exists \( z_0 \in \mathbb{D} \) such that \( f(z_0) = 0 \).

b) Let \( f : \mathbb{D} \to \mathbb{C} \) be a continuous map such that
\[
\frac{f(z)}{z} \in \mathbb{C} \setminus (-\infty, 0] \quad \text{for all} \ z \in \partial \mathbb{D}.
\]
Show that \( f \) has a zero in \( \mathbb{D} \).

c) Use b) to prove the following 2-dimensional version of Brouwer’s Fixed-Point Theorem: if \( g : \mathbb{D} \to \mathbb{D} \) is a continuous map, then \( g \) has a fixed point, i.e., there exists \( z_0 \in \mathbb{D} \) such that \( g(z_0) = z_0 \).

Problem 2:

a) Let \( f : \overline{\mathbb{D}} \to \mathbb{C} \) be a continuous map, and
\[
A_k = \{ e^{iu} : \pi(2k-1)/4 \leq u \leq \pi(2k+1)/4 \} \subseteq \partial \mathbb{D}
\]
for \( k \in \{0, 1, 2, 3\} \). Suppose that \( \text{Re}(e^{ik}f(z)) > 0 \) whenever \( z \in A_k, k \in \{0, 1, 2, 3\} \). Show that \( f \) has a zero in \( \mathbb{D} \).

Hint: First find a simple function \( f \) that satisfies the hypotheses.

b) Consider the square \( Q := [-1, 1] \times [-1, 1] \subseteq \mathbb{R}^2 \cong \mathbb{C} \), and let \( \alpha : [-1, 1] \to Q \) and \( \beta : [-1, 1] \to Q \) be paths in \( Q \). Suppose that \( \alpha \) joins the left and the right side of \( Q \), and that \( \beta \) connects the bottom and top side of \( Q \) (i.e., \( \text{Re} \alpha(\pm 1) = \pm 1 \) and \( \text{Im} \beta(\pm 1) = \pm 1 \)). Show that \( \alpha \) and \( \beta \) have a point in common, i.e., \( \alpha^* \cap \beta^* \neq \emptyset \).

Problem 3: Let \( U \subseteq \mathbb{C} \) be an open set, and \( f : U \to \mathbb{C} \) a holomorphic function. For \( (z, w) \in U \times U \subseteq \mathbb{C}^2 \) define
\[
g(z, w) = \begin{cases} \frac{f(z) - f(w)}{z - w}, & z \neq w, \\ f'(z), & z = w. \end{cases}
\]

a) Show \( g \) is continuous on \( U \times U \).

b) Show that for fixed \( w \in U \) the function \( z \mapsto g(z, w) \) is holomorphic on \( U \).

p.t.o.
Problem 4:

a) Let \( f : \partial D \to \mathbb{C}^* \) be a continuous map, and define \( \gamma(s) = f(e^{2\pi is}) \) for \( s \in [0, 1] \). Show that if \( f(-z) = -f(z) \) for all \( z \in \partial D \), then \( \text{ind}_\gamma(0) \neq 0 \).

b) Prove the 2-dimensional version of the Borsuk-Ulam Theorem: if \( g : \mathbb{S}^2 \to \mathbb{C} \) is a continuous map on the unit sphere \( \mathbb{S}^2 \subseteq \mathbb{R}^3 \), then there exists a pair of antipodal points on \( \mathbb{S}^2 \) with the same image under \( g \); in other words, there exists a point \( p \in \mathbb{S}^2 \) such that \( g(p) = g(-p) \).

Hint: It is convenient to use complex coordinates and make the identification \( \mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R} \) and \( \mathbb{S}^2 = \{(z, h) \in \mathbb{C} \times \mathbb{R} : |z|^2 + h^2 = 1\} \). Argue by contradiction, and consider \( z \in \partial D \mapsto f(z) := g(z, 0) - g(-z, 0) \).