Problem 1: Let \( f : \mathbb{C} \to \mathbb{C} \) be a non-constant entire function.

(a) For \( r > 0 \) and \( w \in \mathbb{C} \) let \( n(r, w) \) be number of times the function \( f \) attains the value \( w \) in the disk \( B(0, r) \) counted with multiplicities (i.e., \( n(r, w) \) is the number of zeros of the map \( z \mapsto f(z) - w \) in \( B(0, r) \) counted with multiplicities). Show that for fixed \( r > 0 \) the function \( w \mapsto n(r, w) \) is a bounded lower semicontinuous function on \( \mathbb{C} \). (6 pts)

(b) Show that for fixed \( r > 0 \) the function \( w \mapsto n(r, w) \) is integrable on \( \mathbb{C} \) and
\[
\int_{B(0,r)} |f'(z)|^2 \, dA(z) = \int_{\mathbb{C}} n(r, w) \, dA(w).
\]
Hint: Express both integrals as path integrals! (6 pts)
Problem 2: Let $\Omega \subseteq \mathbb{C}$ be a region, and $h: \Omega \to \mathbb{R}$ be a harmonic function. Show that if $h$ vanishes on a set of positive measure $M$ in $\Omega$, then $h \equiv 0$. Hint: One way to prove this is to consider the gradient $\nabla h$ of $h$. (12 pts)
**Problem 3:** If $f$ is a non-constant holomorphic function on $D$, we denote by $L_f \in (0, \infty]$ the radius of the “largest” disk contained in the image $f(D)$; more precisely,

$$L_f := \sup \{ r > 0 : \text{there ex. } z_0 \in \mathbb{C} \text{ with } B(z_0, r) \subseteq f(D) \}.$$ 

The number

$$L := \inf \{ L_f : f \in H(D) \text{ and } f'(0) = 1 \}$$

is known as Landau’s constant. Its precise numerical value is not known. The purpose of this problem is to show that $L > 0$ by establishing an explicit positive lower bound for $L$.

(a) Let $\mathcal{B}$ be the family of all functions $f \in H(D)$ satisfying $f(0) = 0$, $f'(0) = 1$, and

$$|f'(z)| \leq \frac{1}{1 - |z|^2} \text{ for } z \in D.$$ 

Show that $L = \inf \{ L_f : f \in \mathcal{B} \}$. Hint: First show that in the definition of $L$ we may assume that the functions $f$ are holomorphic in an open set containing $\overline{D}$. For such a function consider a point $z_0 \in D$, where $|f'(z)|(1 - |z|^2)$ attains a maximum on $D$, and precompose $f$ by a suitable map that sends 0 to $z_0$. (4 pts)

(b) Let $g : D \to \mathbb{C}$ be a holomorphic function with $g(0) = w_0 \geq 0$ and $|g(z)| \leq 1$ for $z \in D$. Show that

$$\text{Re}(g(z)) \geq \frac{w_0 - |z|}{1 - w_0|z|}$$

for all $z \in D$ with $|z| \leq w_0$. (4 pts)

(c) Find an explicit number $c > 0$ such that $B(0, c) \subseteq f(D)$ for all $f \in \mathcal{B}$. (4 pts)

(d) Find an explicit positive lower bound for $L$. (1 pt)
Problem 4: The purpose of this problem is to give an alternative proof of Picard’s Theorem.

(a) By using Problem 3 show that every non-constant entire function $f$ contains arbitrarily large disks in its image, i.e.,

$$L_f = \sup \{ r > 0 : \text{there ex. } z_0 \in \mathbb{C} \text{ with } B(z_0, r) \subseteq f(\mathbb{C}) \} = \infty.$$  

(2 pts)

(b) Let $f$ be an entire function omitting the values $-1$ and $1$. Show that then there exists an entire function $g$ such that $f = \cos(g)$. Which values will $g$ omit? Hint: You can construct $g$ explicitly by taking suitable roots and logarithms.  

(6 pts)

(c) By using (a) and (b) show that every entire function $f$ omitting $-1$ and $1$ is constant.  

(5 pts)