22. Characterization of simply connected regions

**Lemma 22.1.** \( \Omega \subset \mathbb{C} \) region.

a) If \( \Omega \) is homeomorphic to \( \mathbb{D} \), then

b) If \( \partial \Omega \) is connected, then \( \hat{\mathbb{C}} \setminus \Omega \) is connected.

**Proof.** a) Why \( \Omega \approx \mathbb{C} \). Then \( \Omega \) is simply connected (every loop is null-homotopic), and so by the Riemann Mapping Theorem, there exists a conformal map \( f : \mathbb{D} \to \Omega \).

Claim \( w \in \overline{\Omega} \) iff there exists a seq. \( \{z_n\} \) in \( \mathbb{D} \) with \( |z_n| \to 1 \) s.t.

\[ y(z_n) \to w, \]

\[ \Rightarrow \text{if } w \in \overline{\Omega} \text{ then } w \in \overline{\Omega}; \]

so ex. \( \{z_n\} \) in \( \mathbb{D} \) s.t.

\[ y(z_n) \to w \text{ why } z_n \to a \in \overline{\mathbb{D}}. \]

WTS \( a \in \overline{\mathbb{D}} \).

Otherwise, \( a \in \mathbb{D} \). Then \( y(a) = w, w \in \Omega \), and \( w \notin \overline{\Omega} \). Contradiction!

\[ \Rightarrow \text{if } w \in \overline{\Omega} \text{ then ex. } \{z_n\} \text{ in } \mathbb{D} \text{ with } |z_n| \to 1 \text{ and } y(z_n) \to w \text{ then } w \in \overline{\Omega}. \]

WTS \( w \in \overline{\Omega} \).

Otherwise, \( w \in \Omega \); then ex. \( a \in \mathbb{D} \) s.t.

\[ y(a) = w. \text{ Then } z_n = y^{-1}(w), \text{ } z_n \to y^{-1}(w) = a \]
3c) $f = f$ whenever $f \in H(\Omega)$ and $\alpha, \beta$ are piecewise smooth paths in $\Omega$ with the same endpoints.

1) Every holomorphic function on $\Omega$ has a primitive.

2) For every harmonic function $u$ on $\Omega$, there exists $f \in H(\Omega)$ s.t. $u = \text{Re } f$.

3) Every harmonic function on $\Omega$ has a harmonic conjugate.

4) Every zero-free holomorphic function on $\Omega$ has a holomorphic logarithm.

5) Every zero-free holomorphic function on $\Omega$ has a holomorphic square root.

6) $\Omega = \mathbb{C}$ or $\Omega$ is conformally equivalent to $\mathbb{D}$.

7) $\Omega$ is homeomorphic to $\mathbb{D}$.

8) $\partial \Omega$ is connected (as a subset of $\mathbb{C}$).

9) $\mathbb{C} \setminus \Omega$ is connected.

10) For all $f \in H(\Omega)$ there exists a sequence $\{P_n\}$ of polynomials s.t. $P_n \to f$ locally uniformly on $\Omega$.

Proof: We'll prove:

$\textbf{a \leftrightarrow b}$: By definition.

$\textbf{b \rightarrow c}$: Follows from Cor. 14.13.

$\textbf{c \rightarrow d}$: Cauchy's integral theorem.
2. **Contradiction!**

Let \( A_n = \{ z \in D : \left| 1 - \frac{1}{n} \right| < |z| < 1 \} \).

From the claim we deduce

\[
\partial \Omega = \bigcup_{n \in \mathbb{N}} \partial \varphi(A_n).
\]

Hence \( \partial \Omega \) is connected as an intersection of a descending seq. of comp. conn. sets.

(b) \( \text{Ext}(\Omega) = \text{set of exterior pts. of } \Omega \).

\[
\hat{\Omega} = \Omega \cup \partial \Omega \cup \text{Ext}(\Omega) \quad \text{(disj. union)}.
\]

Suppose \( \partial \Omega \) is conn. and \( \partial \Omega \subset U \cup V \), where \( U, V \subset \hat{\Omega} \) open.

\( U \cap V = \emptyset \).

WTS \( \partial \Omega \subset U \) or \( \partial \Omega \subset V \).

Since \( \partial \Omega \subset U \cup V \) is conn., \( \partial \Omega \subset U \) or \( \partial \Omega \subset V \), say \( \partial \Omega \subset U \).

\( U' = \Omega \cup \partial \Omega \) open, \( V' = V \setminus \overline{\Omega} \) open,

\[ U \cap V' = \emptyset, \quad U \cap V' \subset \partial \Omega \cup \text{Ext}(\Omega) = \hat{\Omega} \]

So \( V' = \emptyset \); \( \forall \, V \subset \overline{\Omega} \) and \( \forall \, V \cap \text{Ext}(\Omega) = \emptyset \).

We conclude \( U \supset \partial \Omega \cup \text{Ext}(\Omega) = \hat{\Omega} \).

---

**Thm 22.2.** (Main Thm of Classical Complex Analysis).

Let \( \Omega \subset \mathbb{C} \) be a region. TFAE:

a) \( \Omega \) is simply conn.

b) every loop in \( \Omega \) is null-homotopic.

c) every loop in \( \Omega \) is null-homologous.

d) \( \int \gamma(z) \, dz = 0 \) whenever \( \gamma \in H(\Omega) \) and \( \gamma \) is a piecewise smooth loop in \( \Omega \).
\( d \to e: \int \frac{f(z)}{z} \, dz = \int f(z) \, d\bar{z} = 0. \)

e \to f: Proof as in Cor. 15.9.a:

\[ F(z) = \int f(w) \, dw, \]

where a piecewise smooth path from base point \( a, z \). Then \( F \) is well-def.

and \( F' = f. \)

\( f \to g: \) Let \( u \) be harmonic on \( \Omega \), i.e., \( u \in C^2(\Omega), \)

\[ \Delta u = u_{xx} + u_{yy} = 0. \]

Define \( g : = \frac{u_x + i u_y}{\alpha + i \beta} \).

\[ a = \Re g, \quad b = \Im g, \quad \alpha = u_x = u_{xx} = -u_{yy} = b_y, \]

\[ \beta = u_y = u_{yx} = -u_{xx} = -a_x. \]

So \( g \) sol. CR-eqs. Hence \( g \in H(\Omega) \).

By \( h \) how \( \Re h \) \( \in H(\Omega) \) s.l. \( h' = g \).

So \( h' = (\Re h)_x + i (\Im h)_x = (\Re h)_x - i (\Re h)_y = g = u_x - i u_y. \)

So \( \nabla u = \nabla (\Re (h) + c) \) s.e. \( c \in \mathbb{R} \) s.l.

\[ u = \Re (h) + c = \Re (h + c), \quad f \in H(\Omega). \]

\( g \to h: \) \( u \) harmonic on \( \Omega \). By \( g \) \( \overline{u} \) on \( f \in H(\Omega) \)

s.l. \( u = \Re (f) \). Then \( u = \Im (f) \) is

harmonic on \( u \) on \( \Omega \).

\( h \to i: \) Let \( f \in H(\Omega), \) \( f(z) \not\equiv 0 \) for \( z \in \Omega \).

Define \( \alpha : = \log \| f \| = \frac{1}{2} \log \| f^2 \| \in C_0^\infty(\Omega). \)

\[ \Delta u = 4 u_{zz} \quad = 2 \left( \frac{\Re f}{f} \right)_z + 2 \left( \frac{\Im f}{f} \right)_z \]

\[ = 0 + 2 \left( \frac{\partial f}{\partial z} \right)_z = 0. \]
So $u$ is harmonic. By hypothesis, $u$ is a positive function in $H(\Omega)$.

Then $e^u$ is also harmonic.

Let $f = e^{u/2}$, then $|f| = |u|/|u| = 1$,

so $e^u/f$ is constant. (Max. principle).

Ex: $c = e^{iz}$, where $z \in \Omega$.

$e^u/f = e^{iz}$, so $f = e^{-z} = e^{-iz}$

$g = e^{iz}$, so $f = e^{iz}$.

$g^{-1} = e^{-iz} 

\Rightarrow j: f \in H(\Omega)$, $f(z) \to 0$ as $z \to \infty$.

But there exists $g \in H(\Omega)$ such that

$f = e^{ug/2}$. Then $\Omega := e^{ug/2} \in H(\Omega)$ is a

holomorphic square root of $f$.

\rightarrow K: Suppose $\Omega = C$ and every zero-free
holomorphic function has a square root.

Then by the proof of the Riemann Mapping
Theorem, after proof of Thu. 19.51),

$\Omega$ is conformal equivalent to $D$.

\rightarrow l: If $\Omega$ is conformal equivalent to $D$, then $\Omega$
is homeomorphic to $D$ if $\Omega = C$, then $\Omega$ is
also homeomorphic to $D$ (exercise!).

\rightarrow b: Obvious.

\rightarrow m: Loc. 22.1. a).

\rightarrow n: Loc. 22.1. b).

\rightarrow o: Cor. 21.7.

\rightarrow d: Let $f \in H(\Omega)$. Then by o there exists

a sequence of polynomials $\{P_n\}$ such that

$P_n \to f$ loc. uniformly on $\Omega$. If $f$ is a piecewise
smooth loop, then
23. The inhomogeneous Cauchy–Riemann equations

Rem. 23.1. $U \subset \mathbb{C}$ open.

The equation

$$\frac{\partial u}{\partial \overline{z}} = 0 \quad (\star)$$

for an unknown function $u$ and a given function $\alpha$ on $U$ is known as the inhomogeneous Cauchy–Riemann equation, or simply as the $\alpha$-equation. (\alpha_bar).

It is a generalization of the CR-eqs.

(\text{homogeneous case with } u \equiv 0).

If $U \subset \mathbb{C}$ and $u \in C^1(U)$, then (\star) has a solution $f \in C^1(U)$, namely $f = T_u$ (Cauchy transform).

Indeed, by Lem. 20.3. $(T_u)^z = u$.

Thm. 23.2. Let $U \subset \mathbb{C}$, and $u \in C^1(U)$.

Then there exist $f \in C^1(U)$ s.t.

$$\frac{\partial f}{\partial \overline{z}} = u \quad (\star)$$

If $f \in C^1(U)$ is another solution of (\star),
7) Then there is $h \in H(U)$ s.t.
$$f = f + h.$$ 

Proof: Uniqueness clear.

Existence: Let $\{K_n\}$ be a comp. exhaustion of $U$ as in Lem. 21.5. and $A \subseteq C \setminus U$ be a set that meets each bold comp. comp. of $C \setminus U$ (pick a point in each!). By Lem. 20.7, there exist $\phi_n \in C^\infty_c(U)$ s.t. $\phi_n|_{K_n} \equiv 1$.

Let $u_n := \phi_n \cdot u \in C^\infty_c(C)$ (extend by 0 outside $U$ as usual). Then $g_n := T u_n \in C^1(U)$ and $\frac{g_{n+1} - g_n}{z} = (T u_n) \frac{1}{z} = u_n \mid_{K_n}, n \in \mathbb{N}.$

Note 1: $g + \sum_{n=1}^\infty (u_{n+1} - u_n) = \lim_{n \to \infty} u_n = u$ p-wise on $U$.

To solve (1), we would like to set 
$$f := g + \sum_{n=1}^\infty (g_{n+1} - g_n),$$ but this series will not converge in general.

Wont: to correct series by holomorphic terms to make it convergent.

Note
$$\frac{g_{n+1} - g_n}{z} = u_{n+1} - u_n = (\phi_{n+1} - \phi_n) \cdot u \equiv 0 \text{ on } K_n.$$

So $g_{n+1} - g_n$ is holomorphic on int $(K_n) \supseteq K_{n-1}, K_0 = \emptyset$.

By construction of the exhaustion, $A$ meets
By the improved upper lemma (Lem. 21.4), there exists a rational function $R_n$ with no poles outside $\text{Aut} \log S_t$.

$$|g_{n+1} - g_n - R_n| < \frac{1}{2^n} \quad \text{on } K_{n-1}, n \in \mathbb{N}.$$ 

In particular, $R_n$ is holomorphic on $U$.

Define

$$f = g_1 + \sum_{n=1}^{\infty} (g_{n+1} - g_n - R_n)$$

By the Weierstrass $M$-test, this series converges uniformly on each set $K_N$.

$$f = g_1 + \sum_{n=1}^{N-1} (g_{n+1} - g_n - R_n) + \sum_{n=N+1}^{\infty} (g_{n+1} - g_n - R_n)$$

$$|h_n| \leq \frac{1}{2^n} \quad \text{for } n = N+1, h_n \in H(\text{int}(K_N)).$$

Hence it converges locally uniform on $U$.

Moreover, we have

$$f = g_1 + \sum_{n=1}^{N-1} (g_{n+1} - g_n - R_n) + h_N \quad \text{on } \text{int}(K_N),$$

where $h_N \in H(\text{int}(K_N))$.

This shows that $f$ is $C^\infty$ smooth on $U$, and

$$D_{\mathbb{C}} f = \frac{\partial f}{\partial \zeta} = \frac{\partial g_{N+1}}{\partial \zeta} = g_{N+1} \cdot u = u \quad \text{on } \text{int}(K_N).$$

Hence $\frac{\partial f}{\partial \zeta} = u$ on $U$. $\square$
24. Partial fraction decompositions
of meromorphic functions
(Mittag-Leffler)

Thm. 24.1. Let $U \subseteq \mathbb{C}$ be open, $A \subseteq U$ be a discrete set (equiv. $A \cap U$ has no limit pts. in $U$).

Suppose for each $a \in A$ we specify a function of the form

$$P_a(z) = \sum_{n=1}^{\infty} \frac{A_n(a)}{(z-a)^n}, \quad n(a) \in \mathbb{N},$$

then there exists a meromorphic function $f \in H(U)$ on $U$ that has no poles outside $A$, and that has a pole with principal part $P_a$ for each $a \in A$.

Proof: Pick a compact exhaustion $K_n, n \in \mathbb{N}$, as in Lem. 21.5.

Define $A_n := K_n \setminus K_{n-1}, n \in \mathbb{N}$, where $K_0 := \emptyset$.

Then $A_n$ is finite, and we have a disjoint union $A = \bigcup_{n \in \mathbb{N}} A_n$.

Define $g_n := \sum_{a \in A_n} P_a$.

Then $g_n$ is meromorphic on $U$, and holomorphic on $\text{int} (K_{n-1})$ (there are poles in $K_n$).

By the improved approximation lemma (Lem. 21.4), there exists $R_n \in H(U)$ (see the proof of Thu. 23.2) s.t.

$$|g_n - R_n| < \frac{1}{2^n} \quad \text{on} \quad K_{n-2} \setminus \text{int} (K_{n-1}).$$
10. Define \( f = g_1 + \sum_{n=2}^{\infty} (g_n - R_n). \)

If \( N \in \mathbb{N} \) then on \( K_N \) we have
\[
f = g_1 + \sum_{n=2}^{N+1} (g_n - R_n) + \sum_{n=N+2}^{\infty} (g_n - R_n).
\]

Note that in the last series each term is holomorphic and bounded by \( 1/2^n \) on \( K_N \cap K_{N-1} \).

Hence this series represents a holomorphic function \( H_N \) on \( \text{int} (K_N) \).

So \( f = g_1 + \sum_{n=2}^{N+1} (g_n - R_n) + H_N \) on \( \text{int} (K_N) \).

which shows that \( f \) is holomorphic on \( \text{int} (K_N) \) and that \( H_N \) has poles with principal part pa for \( a \in A \cap \text{int} (K_N) \).

Since \( \{U_n \cap \text{int} (K_N) \} = U \), the claim follows.

\[\text{Rev. 29.2.} \quad \text{If} \quad \sum_{n=1}^{\infty} h_n \quad \text{is a series of holomorphic functions} \quad h_n \quad \text{that converges locally uniformly on an open set} \quad U, \]
\[\text{then it represents a holomorphic function} \quad H \quad \text{and we can differentiate term-by-term to arbitrary order} \quad K \in \mathbb{N}: \]
\[H^{(k)}(z) = \sum_{n=1}^{\infty} h_n^{(k)} \quad \text{on} \quad U, \quad \text{with loc. unif. convergence.} \]
This follows from Weierstrass Thm. 18.4, applied to the partial sums of the series.

In the previous proof we represented $f$ locally as a finite sum of meromorphic functions plus a locally uniformly convergent series of holomorphic functions.

It follows that we can differentiate term-by-term to arbitrary order.

**Ex. 24.3.** $f(z) = \frac{\pi^2}{\sin^2 \pi z}$.

Poles for $n \in \mathbb{Z}$:

$$\sin n = n - \frac{n^3}{6} + \cdots \quad n \to 0$$

$$\frac{\pi^2}{\sin^2 \pi z} = \frac{\pi^2}{\sin^2 \pi(z-n)} = \frac{\pi^2}{(\pi (z-n) - \frac{1}{6} \pi^3 (z-n)^3 + \cdots) + \cdots}$$

$$= \frac{1}{(z-n)^2} \left(1 - \frac{1}{3} \pi^2 (z-n)^2 + \cdots\right)$$

$$= \frac{1}{(z-n)^2} \left(1 + \frac{1}{3} \pi^2 (z-n)^2 + \cdots\right)$$

So, principal part $p_n(z) = \frac{1}{(z-n)^2}$.

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \quad \text{conv. loc. unif. \quad \text{on } \mathbb{C} \setminus \mathbb{Z}}$$

$K = B(0, R)$; for $z \in B(0, R)$, $|n| \geq 2R$,

$$|z| \leq |n|/2 \Rightarrow$$

$$\left| \frac{1}{(z-n)^2} \right| \leq \left| \frac{1}{|n|^2} \right| \leq \frac{4}{|n|^2}$$

$$\sum_{|n| \geq 2R} \frac{1}{|n|^2} \leq 4 \sum_{|n| \geq 2R} \frac{1}{|n|^2} \leq 0 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$
\[ f(z) = \sum_{|n| < 2R} \frac{1}{(z-n)^2} + \sum_{|n| \geq 2R} \frac{1}{(z-n)^2} \] is holomorphic on \( B(0, R) \). Hence \( \hat{f} \) is holomorphic on \( \mathbb{C} \). poles for \( n \in \mathbb{Z} \), principal part \( \frac{1}{(z-n)^2} \). 

\[ \hat{f}(z+1) = \hat{f}(z), \text{ indeed} \]

\[ \hat{f}(z+1) = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{(z-(n-1))^2} \]

\[ = \lim_{N \to \infty} \sum_{n=-N}^{N-1} \frac{1}{(z-n)^2} = \]

\[ = \lim_{N \to \infty} \left[ \sum_{n=-N}^{N-1} \frac{1}{(z-n)^2} + \frac{1}{(z+N+1)^2} \right] = \hat{f}(z). \]

\( S = \{ x+iy : 0 \leq x \leq 1, y \in \mathbb{R} \} \).

For \( x+iy \in S \), \( |y| = 2, \) \( x \in [0, 1] \):

\[ \left| \frac{1}{(z-n)^2} \right| = \frac{1}{(n-x)^2 + y^2} \leq \frac{1}{y^2} \frac{1}{1 + \frac{(n-x)^2}{y^2}} \leq \frac{4}{y^2} \frac{1}{4 + (n-x)^2} = \frac{4}{y^2} \frac{1}{n^2 + 4 - 2nx} \leq \frac{4}{y^2} \frac{1}{(n^2 - 2n + 1)3} = \frac{4}{y^2} \frac{1}{3 + (n-1)^2} \]

\[ |f(z)| \leq \sum_{n=-\infty}^{\infty} \frac{1}{|z-n|^2} \leq \frac{4}{y^2} \sum_{n=-\infty}^{\infty} \frac{1}{3 + (n-1)^2} \]

\[ = \frac{c}{y^2} \]

So \( |\hat{f}(z+y)| \to 0 \) uniformly for \( x+iy \in S \), \( |z| \to \infty \).
Similarly, for \( z = x + iy \in \mathbb{S} \),
\[
|f(z)| = \left| \frac{\pi^2}{\sin^2 \pi z} \right| = \left| \frac{-4\pi^2}{e^{i\pi z} - e^{-i\pi z}} \right|^2 \\
= \frac{4\pi^2}{\left| e^{i\pi z} - e^{-i\pi z} \right|^2} \leq \frac{4\pi^2}{\left| e^{iy} - e^{-iy} \right|^2} \\
= \frac{\pi^2}{\sinh^2 y} \to 0 \text{ as } |y| \to \infty.
\]

Hence \( h(z) = f(z) - \tilde{f}(z) \) is an entire function.
It is 1-periodic, odd, holomorphic on \( \mathbb{S} \) and hence
on \( \mathbb{C} \). So \( h(z) \equiv 0 \) on \( \mathbb{C} \) by Liouville.
Since \( h(z) \to 0 \) as \( z \to \infty \), \( |z| \to \infty \),
it follows that \( h \equiv 0 \).

Conclusion:
\[
\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}, \quad z \in \mathbb{C} \setminus \mathbb{Z}.
\]

Second proof of Milne-Thompson's Theorem:

For each point \( a \in A \) we can pick \( r_a > 0 \)
so \( A \cap B(a, r_a) = \{a\} \).
Define \( B_a := B(a, r_a \sqrt{2}) \). Then \( B_a \cap B_b = \emptyset \)
for \( a \neq b \in A \).
Pick a function \( \varphi_a \in C^\infty_c(\mathbb{C}) \) s.t.
\( \varphi_a \equiv 1 \) near \( a \), \( \text{supp}(\varphi_a) \subseteq B_a \).

Let
\[
g := \sum_{a \in A} \varphi_a \cdot p_a. \quad \text{Then } g \in C^\infty(\mathbb{C} \setminus A),
\]
\[\frac{\partial g}{\partial z} = \sum_{a \in A} \frac{\partial \varphi_a}{\partial z} \cdot p_a \in C^\infty(\mathbb{C} \setminus A) \quad (\text{w.r.t. } \text{Let})
\]
\[
\frac{\partial^2 g}{\partial \overline{z}^2} \equiv 0 \quad \text{w.r.t. } \text{Let}
\]

Let \( h \in C^1(\mathbb{C}) \) be a solution of
14 \[ \frac{2h}{h} = a \] (exists by Thm. 23.2).
Then \( f = g - h \) is holomorphic on \( U \setminus A \), because 
\( f \in C(U \setminus A) \) and 
\[ \frac{\partial f}{\partial z} = \frac{\partial g}{\partial z} - \frac{\partial h}{\partial z} = 0. \]
Since \( \frac{\partial h}{\partial z} \equiv 1 \) near \( a \), \( a \equiv 0 \) near \( a \), and so \( h \) is holomorphic near \( a \).
Hence near \( a \):
\[ f = g - h = p_n + \text{holomorphic function}. \]
So \( f \) has a pole at \( a \) with principal part \( p_n \).

Ex. 24.4. \( f(z) = \prod \cot \frac{\pi z}{n} = \prod \frac{\cos \frac{\pi z}{n}}{\sin \frac{\pi z}{n}}. \)

Poles for \( a \in \mathbb{Z} \):
\[ \prod \cot \frac{\pi z}{n} = \prod \cot \varphi(z-n) = \prod \frac{\cos \frac{\pi (z-n)}{n}}{\sin \frac{\pi (z-n)}{n}} \]
\[ = \prod \frac{1 - \frac{1}{2} \frac{n^2}{(z-n)^2} + \cdots}{\frac{n}{(z-n)} - \frac{1}{6} \frac{n^3}{(z-n)^3} + \cdots} \]
\[ = \frac{1}{(z-n)} \left( 1 + \cdots \right). \]

So, principal part \( q_n(z) = \frac{1}{z-n} \).
We'd like to consider
\[ \sum_{n = -\infty}^{\infty} \frac{1}{z-n}, \text{ but this series diverges.} \]
(Cess. harmonic series!)
Need to subtract holomorphic continuous term: \( z \in \mathbb{C} \) fixed, \( n \to y, \) \( |y| > |z| \).
\[ \int_{y}^{y+1} \frac{1}{z-n} = \frac{1}{y} - \frac{1}{z-n} = -\frac{1}{y} \left( 1 + \frac{z}{n} + \frac{z^2}{n^2} + \cdots \right) \]
Subtract partial sum of this series.
Here first term is enough:
\[ \hat{f}(z) = \frac{1}{z} + \sum_{n \in \mathbb{Z} \setminus \{0\}} \left( \frac{1}{z-n} + \frac{1}{n} \right). \]  \hspace{1cm} (\ast)

Conv. loc. unif. on \( \mathbb{C} \setminus \mathbb{Z} \).
\( K = B(0, R) \); so \( z \in B(0, R) \), \( |n| \geq 2R \),

\[ \left| \frac{1}{z-n} + \frac{1}{n} \right| = \frac{|z|}{|z-n| \cdot |n|} \leq \frac{2R}{|n|^2} , \]

and
\[ \sum_{|n| \geq 2R} \left| \frac{1}{z-n} + \frac{1}{n} \right| \leq \frac{2R}{2} \sum_{|n| = 2R} \frac{1}{|n|^2} = \infty . \]

This shows that \( \hat{f} \) is meromorphic on \( \mathbb{C} \setminus \mathbb{Z} \),

and has poles \( \sum \frac{1}{n} \) with principal

poles \( \mathbb{Z} \) with principal

poles \( \mathbb{Z} \) with principal

poles \( \mathbb{Z} \).

We can differentiate (\ast) term-by-term:
\[ \hat{f}'(z) = -\frac{1}{z^2} + \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{(z-n)^2} = -\sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{(z-n)^2} \]

\[ = -\frac{\pi^2}{\sin^2 \pi z} \quad (\text{Ex. 24.3}). \]

\[ f'(z) = \frac{d}{dz} \left( \pi \frac{\cos \pi z}{\sin \pi z} \right) = \pi \frac{\sin^2 \pi z - \cos^2 \pi z}{\sin^2 \pi z} \]

\[ = -\frac{\pi^2}{\sin^2 \pi z} . \]

So \( h = f - \hat{f} \) is entire function with \( h' \equiv 0 \).

Hence \( h \equiv \text{const.} \).

Note that \( f \) and \( \hat{f} \) are odd functions;

Hence \( f \) is odd.

Since \( f(0) = 0 \),

\[ \hat{f}(-z) = -\frac{1}{z} + \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{z-n} + \frac{1}{n} = -\left( \frac{1}{z} + \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{z-n} + \frac{1}{n} \right) = \hat{f}(z) , \]
16. Hence $h$ is count. and odd, and so $h \equiv 0$.

**Conclusion**

\[ \cot n z = \frac{1}{z} + \sum_{n \in \mathbb{Z} \setminus \{0\}} \left( \frac{1}{z - n} + \frac{1}{n} \right), \quad z \in \mathbb{C} \setminus \mathbb{Z}. \]

Alternatively,

\[ \cot n z = \frac{1}{2} \left( \frac{1}{z} + \sum_{n \in \mathbb{Z} \setminus \{0\}} \left( \frac{1}{z - n} + \frac{1}{n} \right) \right) \]

\[ + \frac{1}{2} \left( \frac{1}{z} + \sum_{n \in \mathbb{Z} \setminus \{0\}} \left( \frac{1}{z + n} - \frac{1}{n} \right) \right) \]

\[ = \frac{1}{z} + \frac{1}{2} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{2z}{z^2 - n^2} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}. \]

25. Infinite products

**Def.** 25.1. \{an\} seq. in C.

Define

\[ p_n = a_1 \cdots a_n = \prod_{k=1}^{n} a_k. \]

\( p_n \) is repr. by symbol \( \prod_{n=1}^{\infty} a_k \), called an **infinite product**.

\( p_n \) partial product.

We say that \( \prod_{n=1}^{\infty} a_n \) converges (properly) if there ex. \( n_0 \in \mathbb{N} \) s.t. the limit \( \lim_{n \to \infty} \prod_{k=n_0}^{n} a_k \) exists and is non-zero.

Then \( \lim_{n \to \infty} p_n = \lim_{n \to \infty} \prod_{k=1}^{n} a_k \) also exists.
and we denote this limit by 
\[ \lim_{n \to \infty} a_n. \]

**Lemma 2.5.2.** Let \( a_n \in C \) for \( n \in \mathbb{N} \).

a) If \( \prod_{n=1}^{\infty} a_n \) converges, then \( a_n = 0 \) for all but finitely many \( n \in \mathbb{N} \), and \( \prod_{n=1}^{\infty} a_n = 0 \) if there exists \( n \in \mathbb{N} \) with \( a_n = 0 \).

b) (Cauchy criterion)
\[ \prod_{n=1}^{\infty} a_n \text{ converges if and only if for all } \epsilon > 0 \]
there exist \( N, N' \in \mathbb{N} \) s.t. for all \( m \geq n \geq N \)
we have
\[ \left| \prod_{k=n}^{m} a_k - 1 \right| < \epsilon. \]

c) If \( \prod_{n=1}^{\infty} a_n \) converges, then \( \lim_{n \to \infty} a_n = 1. \)

**Proof.**

a) Clear from def.

b) \( n \to n' \): Wlog \( \prod_{n=1}^{\infty} a_n \neq 0. \) Then \( a_n \neq 0 \),
\[ a_n = \prod_{k=1}^{n} a_k \to L \neq 0. \]
So \( |a_n| = |L|/2 \) for \( n \) large, and
\[ \left| \prod_{k=n}^{m} a_k - 1 \right| = \left| \frac{a_m}{a_{m-1}} - 1 \right| \leq \frac{1}{|a_{m-1}|} |a_m - a_{m-1}| \leq \frac{2}{|L|} |a_m - a_{m-1}|. \]
The claim follows from Cauchy crit. for the seq. \( \{a_n\}. \)
By considering a "tail" of the inner product, we may assume that
\[ |p_n - 1| \leq \frac{1}{2} \text{ and so } \frac{1}{2} \leq |p_n| \leq 2 \quad (*) \]
for all \( n \in \mathbb{N} \).

Then for \( m = n \geq 2 \),
\[ |p_m - p_n| = |p_{n+1} - 1| \leq 2 |p_{n+1} - 1| \]

Hence, \( \{p_n\} \) is a Cauchy sequence and converges. Moreover,
\[ \lim_{n \to \infty} p_n = 0 \quad \text{by (*)}. \]

(1) follows from (b): pick \( m = n \).

Often one writes an infinite product in the form
\[ \prod_{n=1}^{\infty} (1 + c_n). \]

Necessary for convergence: \( c_n \to 0 \) as \( n \to \infty \), but not sufficient. (\( c_n \) have to go to 0 fast enough!)

**Def. 25.3.** Consider an infinite product of the form
\[ \prod_{n=1}^{\infty} (1 + b_n), \quad b_n \in \mathbb{C}, \quad n \in \mathbb{N}. \]

We say that it converges absolutely if
\[ \prod_{n=1}^{\infty} (1 + |b_n|) \text{ converges.} \]
9. \( \text{Ex. 25.4. a) } \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) \text{ does not converge} \)

\[ p_n = \prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) = \prod_{k=1}^{n} \left(\frac{k+1}{k}\right) = \frac{2}{1} \cdot \frac{3}{2} \cdots \frac{n+1}{n} \]

"telescoping product" \( \rightarrow n+1 \rightarrow \infty \)

b) \( \prod_{n=1}^{\infty} \left(1 + \frac{(-1)^{n-1}}{n}\right) \) converges:

\[ 1 + \frac{(-1)^{n-1}}{n} = \frac{n + (-1)^{n-1}}{n} = \begin{cases} \frac{n+1}{n} & \text{odd} \\ \frac{n-1}{n} & \text{even} \end{cases} \]

\[ p_n = \prod_{k=1}^{n} \left(1 + \frac{(-1)^{k-1}}{k}\right) = \frac{2}{1} \cdot \frac{4}{3} \cdot \frac{3}{4} \cdots \frac{n}{n-1} \]

\[ = \begin{cases} 1 & \text{odd} \\ \frac{n-1}{n} & \text{even} \end{cases} \rightarrow 1 \text{ as } n \rightarrow \infty. \]

By a) the int. prod. does not conv. absolutely.

Prop. 25.5. \( \forall n \in \mathbb{N}, \) \( b_n \in \mathbb{R} \), \( n \in \mathbb{N} \),

a) If \( \prod_{n=1}^{\infty} (1 + b_n) \) converges absolutely, then it converges.

b) \( \prod_{n=1}^{\infty} (1 + b_n) \) converges absolutely if and only if \( \sum_{n=1}^{\infty} |b_n| \) converges.

c) Suppose \( \prod_{n=1}^{\infty} (1 + b_n) \) converges absolutely and let \( \phi: \mathbb{N} \rightarrow \mathbb{N} \) be any bijection. Then \( \prod_{n=1}^{\infty} (1 + b_{\phi(n)}) \) converges absolutely and

\[ \prod_{n=1}^{\infty} (1 + b_{\phi(n)}) = \prod_{n=1}^{\infty} (1 + b_n) \]

(So in an absolutely conv. int. prod. we...
(20) can reorder the factors in any way without changing the convergence behavior or the limit.

Proof: \( p_{n,m} = \prod_{k=n}^{m} (1 + b_k) \), \( q_{n,m} = \prod_{k=n}^{m} (1 + |b_k|) \) for \( m \geq n \). Then

\[ |p_{n,m} - 1| \leq |q_{n,m} - 1| \text{ for } m \geq n. \] (†)

Indeed,

\[ |p_{n,m} - 1| = \left| \frac{(1 + b_n) \cdots (1 + b_m)}{1 + b_n \cdots + b_m} - 1 \right| = \left| \frac{b_n \cdots b_m}{1 + b_n \cdots + b_m} \right| = \sum \left| b_k \right| \cdots \sum \left| b_k \right| \left| b_k \right| + \cdots
\]

\[ = (1 + |b_n|) \cdots (1 + |b_m|) - 1 = |q_{n,m} - 1|. \]

The claim follows from the Cauchy criterion for infinite products.

b) \( s_{n,m} = \sum_{k=n}^{m} |b_k| \). Note \( 1 + x = e^x \) for \( x \geq 0 \).

Then \( |s_{n,m}| \leq |q_{n,m} - 1| \leq e - 1 \) for \( m \geq n \).

Indeed,

\[ |s_{n,m}| = \sum_{k=n}^{m} |b_k| \leq (1 + |b_n|) \cdots (1 + |b_m|) - 1
\]
\[ \leq e^{|b_n|} \cdots e^{|b_m|} - 1 = e - 1 \]

Using the Cauchy criterion for convex infinite series and products, the first inequality in (††)
(2) shows that \( \sum_{n=1}^{\infty} |b_n| \) converges, so \( \prod_{n=1}^{\infty} (1+b_n) \) converges. The second inequality shows the converse (note that \( e-1 \) is small for \( \varepsilon > 0 \) small).

c) Absolute convergence of \( \prod_{n=1}^{\infty} (1+b_k(n)) \)
follows from (b) and
\[
\sum_{n=1}^{\infty} |b_k(n)| = \sum_{n=1}^{\infty} |b_n| < \infty
\]

Let \( L := \prod_{n=1}^{\infty} (1+b_n) \), \( P_n := \prod_{k=1}^{n} (1+b_k) \)
\( \Phi_m := \prod_{k=1}^{m} (1+b_k) \).

Let \( \varepsilon > 0 \), \( \varepsilon \Log \varepsilon = 1 \).

We can pick \( N \in \mathbb{N} \) so, \( \Log \), that
\[
|P_n - L| < \varepsilon \quad \text{and} \quad \sum_{k=N+1}^{\infty} |b_k| < \varepsilon.
\]

Choose \( M \in \mathbb{N} \) so,
\[
\{ \Phi(1), \ldots, \Phi(M) \} \supseteq N_{3}\cup E.
\]

Then for \( m \geq M \)
\[
\{ \Phi(1), \ldots, \Phi(m) \} = \{ \Phi(1), \ldots, N_{3}\cup E \}
\]

Hence for \( m \geq N \)
\[
|P_m - L| \leq |P_m - P_N| + |P_N - L|,
\]
\[
\leq |P_N| \cdot \left| \prod_{k \in E} (1+b_k) - 1 \right| + \varepsilon,
\]
\[
\leq (L+1) \cdot \left( \prod_{k \in E} (1+|b_k|) - 1 \right) + \varepsilon,
\]
\[
\leq (L+1) \cdot (e^{\sum_{k=1}^{\infty} |b_k|} - 1) + \varepsilon.
\]

As \( \varepsilon \to 0 \), the above expression converges to
\[
(L+1)(e-1) + \varepsilon \leq C \varepsilon.
\]
22) Hence \(\lim_{n \to \infty} \prod_{n=1}^{\infty} (1 + b_n(z)) = \lim_{n \to \infty} f_n(z) = 1.\)

**Thm. 25.6. (Infinite products of holomorphic functions)**

Let \( U \subset \mathbb{C} \) open, \( f_n \in H(U) \) for \( n \in \mathbb{N} \).

Suppose that

\[
\sum_{n=1}^{\infty} |f_n(z) - 1| \text{ converges locally uniformly on } U.
\]

Then

\[
f(z) = \prod_{n=1}^{\infty} f_n(z)
\]

converges absolutely and locally uniformly on \( U \) (i.e., \( f_n \to f \) locally uniformly on \( U \) for the partial products).

In particular, \( f \) is holomorphic on \( U \).

**Proof:**

\[
\prod_{n=1}^{\infty} f_n(z) = \prod_{n=1}^{\infty} \left[ 1 + (f_n(z) - 1) \right]
\]

absolutely converges on each \( z \in U \) follows.

We'll establish compact convergence.

Let \( K \subset U \) be compact.

Then

\[
\sum_{n=1}^{\infty} |f_n(z) - 1| \text{ converges uniformly on } K.
\]

Then there exists \( C_0 \geq 0 \) s.t.

\[
\sum_{n=1}^{\infty} \sum_{k=1}^{n} |f_k(z) - 1| \leq C_1 = L(z) \leq C_1
\]

for \( z \in K \).
23. Hence for \( n \in \mathbb{N}, z \in \mathbb{K}, \)

\[
|p_n(z)| \leq |1 + |p_n(z)| - 1|
\]

\[
= |1 + \prod_{k=1}^{n} (1 + |f_k(z)| - 1) - 1|
\]

\[
\leq 1 + \prod_{k=1}^{n} (1 + |f_k(z)| - 1) - 1 \quad \text{(see Proposition 26.5)}
\]

\[
\leq \exp \left( \sum_{k=1}^{n} |f_k(z)| - 1 \right) \quad \text{(see Proposition 26.5)}
\]

\[
\leq C_1 = C_2
\]

So, the functions \( p_n \) are unit-bounded on \( \mathbb{K} \).

Let \( \varepsilon > 0 \). be arb. \( \{w_n\}_{n=1}^{\infty} \) \( \varepsilon \leq 1 \).

Since \( \sum_{n=1}^{\infty} |f_n(z)| - 1 \) conv. unif. on \( \mathbb{K} \),

there ex. \( N \in \mathbb{N} \) s.t. \( \sum_{n=N}^{\infty} |f_n(z)| - 1 \leq \varepsilon \)

for all \( z \in \mathbb{K} \).

Then for \( m \geq n \geq N, z \in \mathbb{K}, \)

\[
|p_m(z) - p_n(z)| = |p_n(z)(1 + \prod_{k=n+1}^{m} (1 + |f_k(z)| - 1) - 1)|
\]

\[
\leq C_2 \left[ \exp \left( \sum_{k=n+1}^{m} |f_k(z)| - 1 \right) - 1 \right]
\]

\[
\leq C_2 (e - 1) \leq 3C_2 \varepsilon.
\]

The unif. conv. of \( p_n \to f \) on \( \mathbb{K} \) follows.

Since \( p_n \in H(U) \), we have \( f \in H(U) \).

**Cor. 25.7.** Assume in **Thm. 26.6.**, in addition that no sector \( S \) vanishes identically on any comp. of \( U \).
24. a) Then \( f \) does not vanish identically on any compact \( \overline{U} \), and so
\[
Z(f) := \{ a \in U : f(a) = 0 \}
\]
consists of isolated pts.
a \( \in Z(f) \) if \( f_n(a) = 0 \) for some \( n \in \mathbb{N} \).
Moreover, if \( a \in Z(f) \), then
\[
f_n(a) = 0 \quad \text{for at least finitely many } n \in \mathbb{N},
\]
and the order of the zero \( a \) on \( f \) is the sum of the orders at \( a \) for the functions \( f_n \) with \( f_n(a) = 0 \).

b) We have
\[
\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \frac{f_n'(z)}{f_n(z)} \quad \text{for } z \in U \setminus Z(f),
\]
and the series converges locally uniformly on \( U \setminus Z(f) \).

Proof: a) For each point \( a \in U \) there exists some \( N \in \mathbb{N} \) which is an open neighborhood \( V \subseteq U \) such that
\[
\sum_{n=1}^{\infty} |f_n(z) - 1| < 1 \quad \text{for } z \in V.
\]
In particular, \( f_n(z) \equiv 0 \), and so
\[
g(z) := \prod_{n=N+1}^{\infty} f_n(z)
\]
is a holomorphic function on \( V \) with \( g(z) \equiv 0 \) for \( z \in V \).
Since
\[
f(z) = \prod_{n=1}^{\infty} f_n(z) = f_1(z) \cdots f_N(z) \cdot g(z)
\]
the claim follows.
25. b) By Weierstrass, \( p_n \to f \) \text{ loc. unif. on } U.
So for each \( z \in U \setminus Z(f) \)
\[
\frac{f_n(z)}{f(z)} = \lim_{n \to \infty} \frac{p_n(z)}{p_n(z)} = \lim_{n \to \infty} \left( \frac{f_1(z)}{f_1(z)} + \ldots + \frac{f_n(z)}{f_n(z)} \right)
\]
\[= \sum_{n=1}^{\infty} \frac{f_n(z)}{f_n(z)} \]
and we have ptw. conv. on \( U \).
For loc. unif. conv. we let \( a \in U \setminus Z(f) \),
and pick \( \epsilon > 0 \). \( \overline{B}(a, \epsilon) \subset U \).
Then \( \sum_{n=1}^{\infty} \left| f_n(z) - f(z) \right| \leq 2 \) \( \text{ conv. unif. on } \overline{B}(a,\epsilon) \),
and so \( \left| f_n(z) - f(z) \right| \leq \frac{\epsilon}{2} \) \( \forall z \in \overline{B}(a,\epsilon) \)
\( \forall n \).
Moreover, for some fixed \( C \), we have
\( \left| f_n(z) - f(z) \right| \leq C \left| f_n(z) - f(z) \right| \) \( \text{ on } \overline{B}(a,\epsilon/2) \).
Hence for \( N \) large enough,
\[\sum_{n=N}^{\infty} \left| \frac{f_n(z)}{f_n(z)} \right| \leq 2C \sum_{n=N}^{\infty} \left| f_n(z) - f(z) \right| \leq 2C \sum_{n=N}^{\infty} \left| f_n(z) - f(z) \right| \]
locally unif. conv. of \( \sum_{n=1}^{\infty} \frac{p_n(z)}{p_n(z)} \) on \( U \setminus Z(f) \).

**Ex. 25.f.** \( f(z) = \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) \).
\( f_n(z) = 1 - \frac{z^2}{n^2} \), \( \left| f_n(z) - f(z) \right| = \sum_{n=1}^{\infty} \frac{|z|^2}{n^2} \)
converges loc. unif. to \( f \) by Weierstrass M-test.
So \( f \) is an entire function by Thm. 26.6.
with first order zeros \( z_n \) \( \forall n \in \mathbb{N} \).
By Cor. 26.7.
\[
\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \frac{-2z/n^2}{1 - \frac{z^2}{n^2}} = \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \pi \cot \pi z - \frac{1}{z}, \text{ for } z \in \mathbb{C} \setminus \mathbb{Z}.
\]

Note that \( g(z) = \frac{\sin \pi z}{\pi z} \) also has first order zeros at \( \pm n \) in \( \mathbb{C} \setminus \mathbb{Z} \).

\begin{align*}
\lim_{z \to 0} \frac{\sin \pi z}{\pi z} &= \lim_{z \to 0} \left( 1 - \frac{1}{6} \pi^2 z^2 + \ldots \right) = 1, \\
\lim_{z \to 0} \left( 1 - \frac{1}{6} \pi^2 z^2 + \ldots \right) &= 1,
\end{align*}

\[f'(0) = \sum_{n=1}^{\infty} \left( 1 - \frac{1}{n^2} \right) = 1.
\]

This implies that \( g/t = \pi \cos \pi z - \frac{1}{z} = \pi \cot \pi z - \frac{1}{z} \).

Hence, \( g' = \frac{df}{f} \).

Indeed, \( (g/t)' = \frac{g't - f'g}{f^2} = \frac{g(1 - f'/f)}{f^2} = 0 \).

Since \( f(0) = g(0) = 1 \), it follows that \( f = g \).

**Conclusion**: \( \sin \pi z = \pi z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) \) for \( z \in \mathbb{C} \).
26. Factorizations of holomorphic functions

26.1. "Multiplicative" approximation

Let $K \subseteq \mathbb{C}$ be a set, $A \subseteq \mathbb{C} \setminus K$, and $C^*(K) := \{ f \in C(K) : f(w) \neq 0 \text{ for } w \in K \}$.

We want to approximate function $f \in C^*(K)$ by functions of the form

$$g(w) = R(w) \cdot e^{S(w)} \quad (\ast)$$

where $R, S$ are rational, and $R$ has no zeros or poles outside $A \cup \text{Hol}_f$, $S$ has no poles outside $A \cup \text{Hol}_f$.

So

$$R(w) = \frac{\prod_{k=1}^{n} (w-a_k)^{m_k}}{\prod_{l=1}^{m} (w-b_l)^{m_l}} \quad a_1, \ldots, a_n, b_1, \ldots, b_m \in A$$

is holomorphic on $\mathbb{C} \setminus A' \equiv K$, where $A' \subseteq A$ finite.

$M_A(K) \subseteq C^*(K)$ is the set of those functions $A$-subgroup of $C^*(K)$; $f, g \in M_A(K)$

$$\implies f \cdot g \in M_A(K), 1/f \in M_A(K).$$

Note

$$M_\emptyset(K) = \{ e^{P(w)} : P \text{ polynomial} \}.$$

$M_A(K) \subseteq C^*(K)$ is the set of functions that are uniformly approximable by functions in $M_A(K)$.

So

$A_A(K) = \overline{M_A(K)} = \overline{M_A(K)}$ (closure in $C^*(K)$ with topology induced by sup-norm).
Lemma 26.2. (Pushing zeros and poles)

Let $K$ be a connected component of $C \setminus K$, $a \in V$. Suppose $f$ is a rational function with no zeros and poles outside $V \cup \{w\}$. Then $f \in \mathcal{A}_{\text{aj}}(K)$, more specifically, for all $\varepsilon > 0$ there exist a rational function $S$ with no poles outside $\{a, \infty\}$ s.t.

$$|f(w) - (w-a) e S(w)| < \varepsilon$$

for all $w \in K$.

If $V$ is the unbounded connected component of $C \setminus K$, then actually $f \in \mathcal{A}_{\infty}(K)$, i.e., for all $\varepsilon > 0$ there exist a polynomial $P$ s.t.

$$|f(w) - e P(w)| < \varepsilon$$

for all $w \in K$.

Proof:

$$f(w) = \frac{\prod_{k=1}^{N} (w - b_k)}{\prod_{l=1}^{M} (w - c_l)} b_1 \cdots b_N, c_1 \cdots c_M \in V.$$

So, it suffices to show that each factor lies in $\mathcal{A}_{\text{aj}}(K)$. Let $b \in V$ be such. Pick a path $P$ joining $a$ and $b$, and let $P(a) = a_0, a_1, \ldots, a_M = b$. Then $a_0, a_1, \ldots, a_M \in V$. Therefore, for each $k$,

$$f(w) = \frac{\prod_{k=1}^{N} (w - b_k)}{\prod_{l=1}^{M} (w - c_l)} b_1 \cdots b_N, c_1 \cdots c_M \in V.$$

So, it suffices to show that each factor lies in $\mathcal{A}_{\text{aj}}(K)$. Let $b \in V$ be such. Pick a path $P$ joining $a$ and $b$, and let $P(a) = a_0, a_1, \ldots, a_M = b$. Then $a_0, a_1, \ldots, a_M \in V$. Therefore, for each $k$,
\[ |u_k - u_{k-1}| \leq r = \text{dist} (f^*, K). \]

We write
\[
(W-b) = \left( \frac{W-u_k}{W-u_{k-1}} \right) \cdots \left( \frac{W-u_1}{W-u_0} \right) (W-a). \tag{**}
\]

So, it suffices to show: **Claim** if \( u \in V \) and \( |u-v| < r \leq \text{dist} (f^*, K) \) then \( w \mapsto \frac{w-v}{w-u} \) is unit. approx. on \( K \) by functions of the form \( e^{S(u)} \), where \( S \) is rational and has no poles outside \( \{|0,0,0\} \).

\[ \log (1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n \quad \text{converges loc. and unit. on } \mathbb{D}. \]

Main branch
\[
\frac{w-v}{w-u} = \frac{w-u + u-v}{w-u} = 1 + \frac{u-v}{w-u}
\]

\[ |\frac{u-v}{w-u}| = \frac{|u-v|}{|w-u|} \leq \frac{|u-v|}{r} < 1 \quad \text{for } w \in K. \]

So \( \log \left( \frac{w-v}{w-u} \right) = \log (1 + \frac{u-v}{w-u}) \) is unit. approx. on \( K \) by polynomials in \( \frac{1}{w-u} \) and hence by polynomials in \( \frac{1}{w-a} \) (Ler. 21.3).

Hence
\[ \frac{w-v}{w-u} \text{ is unit. approx. on } K \text{ by functions of the form } e^{S(u)} \text{, where } S \text{ is rational and has no poles outside } \{|0,0,0\}. \]

The first part of the statement follows.
Suppose $V$ is the unbold coup of $C \setminus K$. Then we choose $a \in V$ so that $|a| > R = \sup \{|w| : \omega \in K\}$, and use this in (4).

As in the first part of the proof, we can approximate $\log\left(\frac{w-u_i}{w-u_{i+1}}\right)$ unit on $K$ by polynomials in $\frac{1}{w-u_{i+1}}$, and hence by polynomials in $w$ (push $u_{i+1}$ to $\infty$). So $\frac{w-u_i}{w-u_{i+1}}$ can be approximated by functions of the form $e^{P(w)}$, $P$ polynomial.

Note $$(w-a) = -a\left(1 - \frac{w}{a}\right), \quad \text{and} \quad \left|\frac{w}{a}\right| < \frac{R}{|a|} < 1.$$ So $$\log\left(1 - \frac{w}{a}\right)$$ is unit approx on $K$ by polynomials in $P$; hence $(w-a)$ is unit approx on $K$ by functions of the form $e^{P}$. \[\]"}

Lemma 26.3. (Multiplicative approximation lemma)

Let $T$ open, $K \subseteq T$ coup, $f \in H(C)$, $A \subseteq C \setminus K$ that avoids each bold coup of $C \setminus K$. Assume $f(w) \neq 0$ for all $w \in K$.

Then $f \in A^+(K)$, i.e., for all $\varepsilon > 0$, there exists a function $s(w)$ such that

$$g(w) = R(w) \cdot e^{s(w)}$$

$$|f(w) - g(w)| < \varepsilon$$ for all $w \in K$. \[\]"
Have \( P, S \) are rational, \( P \) has no zeros and poles outside \( \mathcal{A}(\sigma) \). \( S \) has no poles outside \( \mathcal{A}(\sigma) \). If \( C \setminus K \) has no bold. comp., then for all \( e > 0 \) there ex. a polynomial \( P \ s.t. \ g(w) = e \) and \( |f(w) - g(w)| < e \) for all \( w \in K \).

Note: \( \forall y, \ e' = \frac{1}{2m} \leq \min \{|f(w)|: w \in K\} \geq 0 \). Then \( |g(w)| \geq \frac{1}{2} \) for \( w \in K \), and

\[
\left| \frac{f(w)}{g(w)} - 1 \right| \leq \frac{1}{|g(w)|} \left| f(w) - g(w) \right| \leq \frac{e}{m}, \quad e' = \frac{e}{2m},
\]

\( e' \leq e \).

Proof: \( \forall y, \ e = \min \{|f(w)|: w \in K\} \). By the Lm. 29.4. (Improved Approx. Lm.) there ex. a rational function with no poles outside \( \mathcal{A}(\sigma) \) s.t.

\[
|f(w) - P(w)| < \frac{e}{2} \leq m \text{ for all } w \in K.
\]

Then \( P(w) \approx f(w) \) for all \( w \in K \), and so \( P \) has no zeros outside \( C \setminus K \cup \{0\} \).

Hence \( R \) can be decomposed into rational functions

\[ R = R_1 \cdots R_k. \]

So, if \( \mathcal{A} \) has \( k \) bold. comp., then ex. a comp. \( V \) at \( C \setminus K \) s.t. \( R_{\mathcal{A}} \) has no poles or zeros outside \( V \cup \{0\} \). There ex. an \( \in V \) \( k \)

So, \( \mathcal{A} \) bold. comp. \( V \) by
3) Functions of the form 
\[(w-a)^n e^{Su(z)}\], where \(w \in \mathbb{Z}\), \(S\) rational with no poles outside \(\mathbb{D}\).

The first statement follows.

For the proof of the second statement, we use the second part of Lem. 26.2 to approximate each factor \(D_k\) by a function of the form \(e^z + \mathcal{P}\) polynomial.

Thm. 26.4. (Weierstrass)

Let \(U \subseteq \mathbb{C}\) be open, \(Z \subseteq U\) be a discrete set (equiv. \(Z \subseteq U\) has no limit pts. in \(U\)). Suppose \(w : Z \to N\) is a function.

Then there exists a holomorphic function \(f\) on \(U\) s.t. \(f\) has no zeroes outside \(Z\), and for each \(p \in Z\) \(f\) has a zero of order \(w(p)\).

(Proof) Pick a countable exhaustion \(K_n, n \in \mathbb{N}\) of \(U\) as in Lem. 21.5.

Define \(Z_n = K_n \setminus K_{n-1}\), \(n \in \mathbb{N}\), where \(K_0 = \emptyset\).

Then \(Z_n\) is finite, and we have a disjoint union \(Z = \bigcup_{n \in \mathbb{N}} Z_n\).

For each \(n \in \mathbb{N}\) define
\[g_n(z) = \prod_{a \in Z_n} (z-a)^{w(a)}\]

Then \(g_n\) is a polynomial with zeroes at \(a \in \mathbb{A}K\) of order \(w(a)\).
We can pick \( A \in \text{C}^{1\leq n} \) such that each bold coup. at \( C \setminus K_n \), \( n \in \mathbb{N} \). We'd like to define
\[
f = \prod_{n=1}^{\infty} g_n, \quad \text{but in general the inf.}
\]
product will not converge.

The function \( g_n \) is holomorphic on \( U \) and zero-free on \( \overline{K_n} \). Hence, by [Hull, approx. theor.] there exists a solution \( h_n \in H(U) \) with \( h_n(z) \neq 0 \) for all \( z \in U \) such that
\[
\left| \frac{g_n}{h_n} - 1 \right| < \frac{1}{2^n} \quad \text{on } K_{n-1}.
\]

Note that \( g_n \) and \( g_n/h_n \) have the same zeros of the same order.

Define \( f \) as
\[
f = \prod_{n=1}^{\infty} \frac{g_n}{h_n}.
\]
Then \( \sum_{n=1}^{\infty} \left| \frac{g_n}{h_n} - 1 \right| \) converges compactly (and hence loc. unif. on \( U \)) by the Weierstrass M-Test, because
\[
\left| \frac{g_n}{h_n} - 1 \right| < \frac{1}{2^n} \quad \text{on } K_n, \quad n \in \mathbb{N}, \ n \geq N+1.
\]

By Thm. 25.6, \( f \) is holomorphic on \( U \).

By Cor. 25.7, \( f \) has zeros precisely at the points \( a \in \mathbb{Z} \), and the order of the zero at \( a \in \mathbb{Z} \) is \( m(a) \). \( \Box \)
Cor. 26.5: \( U \subseteq \mathbb{C} \) region, \( f \) meromorphic on \( U \). Then there exist \( g, h \in H(U) \), \( h \neq 0 \)
\[ b.1. \quad f = g/h \]

**Proof:** Let \( Z \subseteq \mathbb{C} \) be the set of poles of \( f \), and \( m(p) \) the order of the pole \( p \in Z \). Then \( Z \) has no limit points in \( U \). By Thm. 26.4, there exists \( h \in H(U) \) s.t. \( h(z) \rightarrow 0 \) as \( p \in U \setminus Z \), and so that \( h \) has a zero of order \( m(p) \) for \( p \in Z \). Then \( g := h/f \) is holomorphic on \( U \setminus Z \) and has removable singularities for \( p \in Z \). Hence \( g \) is holomorphic on \( U \), and
\[ f = g/h. \quad \square \]

If \( U \subseteq \mathbb{C} \) is a region, then \( H(U) \) is an integral domain, and the previous cor. implies that the quotient field of \( H(U) \) is the set of holomorphic functions on \( U \).

26.6. Elementary factors

If \( U = \mathbb{C} \) in Thm. 26.4, then one can exhaust \( U \) by closed disks \( K_n \).

Then \( \mathbb{C} \setminus K_n \) has no boundary comp. and one can choose functions \( h_n \) as in the proof of Thm. 26.4. of the form \( a^n \cdot P_n(u) \), where \( P_n \) is a polynomial (cf. Lem. 26.3).

It is possible to do this explicitly based...
Let $n \in \mathbb{N}$, and denote

$$E_n(u) = (1-u) \exp \left( u + \frac{u^2}{2} + \cdots + \frac{u^n}{n} \right).$$

Then

$$|E_n(u) - 1| \leq |u|$$

for all $u \in \overline{D}$.

Proof: 

$$-E_n'(u) = \exp \left( P_n(u) \right) \cdot \frac{P_n'(u)}{(1-u)}.$$

and so $-E_n'(u)$ has a zero of order $n+1$ at $0$, and all its Taylor coeff. at $0$ are non-negative.

Hence

$$1 - E_n(u) = -\int_{0}^{u} E_n'(z) \, dz$$

for $u \in \mathbb{C}$.

and we can interpolate this Taylor series term-by-term, it follows that

$$1 - E_n(u)$$

has a zero of order $n+1$ at $0$, and the Taylor coeff. of this function at $0$ are also non-neg.

Hence

$$Q(u) = \frac{1 - E_n(u)}{u^{n+1}}$$

is an analytic function with

$$Q(u) = \sum_{k=0}^{\infty} a_k u^k,$$

where $a_k \geq 0$ for $k \in \mathbb{N}$.

So for $u \in \overline{D}$,

$$|Q'(u)| \leq Q(1) = \frac{1 - E_n(1)}{1^{n+1}} = 1,$$

and the claim follows.
Prop. 26.7. Let \( \{Z_n\} \) be a sequence of complex numbers with \( Z_n \neq 0 \) for \( n \in \mathbb{N} \), and \( Z_n \to \infty \) as \( n \to \infty \).

If \( \{N_n\} \) is a sequence of numbers \( N_n \in \mathbb{N}_0 \) s.t.
\[
\sum_{n=1}^{\infty} \left( \frac{R}{|Z_n|} \right)^{N_n+1} < \infty \quad \text{for all } R > 0,
\]
then the infinite product
\[
f(z) = \prod_{n=1}^{\infty} E_{N_n} \left( \frac{z}{Z_n} \right)
\]
converges absolutely and loc. uni. on \( \mathbb{C} \).
Moreover, \( f \) is entire and has no zeros outside \( \mathbb{Z} := \{Z_n: n \in \mathbb{N}_0\} \).
For \( a \in \mathbb{Z} \) the function \( f \) has a zero at \( a \) whose order is equal to the number of times \( a \) occurs in \( \{Z_n\} \).
If \( N_n = n - 1 \), then (1) holds.

Proof: This follows immediately from Thm. 25.6. and Cor 25.7.
Note that \( E_{N_n}(z/Z_n) \) is an entire function with one zero of order \( 1 \) at \( Z_n \) and no other zeros. Moreover, if \( R > 0 \) is fixed, then
\[
|Z_n| \geq R \quad \text{for large } n, \quad \text{say, for } n \geq n_0.
\]
Hence
\[
\sum_{n=n_0}^{\infty} \left| E_{N_n}(z/Z_n) - 1 \right| \leq \sum_{n=n_0}^{\infty} \left| \frac{z}{Z_n} \right|^{N_n+1}
\]
\[
\leq \sum_{n=n_0}^{\infty} \left( \frac{R}{|Z_n|} \right)^{N_n+1} < \infty \quad \text{for } z \in \overline{B}(0,R).
\]
Hence
\[
\sum_{n=1}^{\infty} \left| E_{N_n}(z/Z_n) - 1 \right| \text{ converges loc.}
\]
Thm. 26.8. (Weierstrass Factorization Theorem)
Let \( f \) be an entire function, \( f(z) \neq 0 \) at \( z \), each zero \( z_n \) of \( f \) each listed as often as its multiplicity. Then there exist numbers \( m \in \mathbb{N}_0, N \in \mathbb{N}_0 \), and an entire function \( h \) s.t.
\[
f(z) = e^{h(z)} z \prod_{n=1}^{\infty} E_{N_n}(z/z_n) \text{ for } z \in \mathbb{C}.
\]

Proof: Based on Prop. 26.7, we can find numbers \( m \in \mathbb{N}_0, N \in \mathbb{N}_0 \) for \( n \in \mathbb{N} \) s.t.
\[
g(z) = z \prod_{n=1}^{\infty} E_{N_n}(z/z_n) \text{ is an entire function with the same zeros as } f \text{ with the same multiplicities.}
\]
Then \( u(z) = f/g \) is an entire function with no zeros. Hence \( u \) has a holomorphic logarithm \( h \) in \( \mathbb{C} \), i.e., there exist \( h \in H(\mathbb{C}) \) s.t.
\[
u(z) = f/g = e^h. \] The claim follows.

Ex. 26.9. We want to find an entire function with zeros at \( n \in \mathbb{Z} \) of first order.
Let \( \{z_n\} \) be an enumeration of \( \mathbb{Z} \) \&\( h \), and \( N_n = 1 \) for all \( n \in \mathbb{N} \).
Then \( \sum_{n=1}^{\infty} R_n > 0 \)
\[
\sum_{n=1}^{\infty} \left( \frac{R}{|z_n|} \right)^2 = 2 \sum_{n=1}^{\infty} \frac{R^2}{n^2} < \infty.
\]
\[ f(z) = z \cdot \prod_{n=1}^{\infty} \left( 1 - \frac{z}{n} \right) e^{-z/n} \]

is an entire function with simple zeros at \( n \in \mathbb{Z} \). Note that the order of the factors in \((\star)\) is irrelevant, as the product converges absolutely. Hence

\[ f(z) = z \cdot \prod_{n=1}^{\infty} \left( 1 - \frac{z}{n} \right) e^{-z/n} \cdot \left( 1 + \frac{z}{n} \right) e^{-z/n} \]

\[ = z \cdot \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) = \frac{\sin \pi z}{\pi z} \quad \text{(cf. 25.8)}. \]

**Ex. 26.10.** Want to construct an entire function \( f \) with simple zeros at the points \(-n, n \in \mathbb{N}_0\).

Take \( a_n = -n \) and \( N_n = 1 \) in Prop. 26.7.

Then \( \sum_{n=1}^{\infty} \frac{R^2}{n^2} < \infty \) for all \( R > 0 \).

So,

\[ f(z) = z \cdot \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right) e^{-z/n} \]

is an entire function with simple zeros at \(-n, n \in \mathbb{N}_0\).

Since the \( \Gamma \)-function has simple poles at those points, we expect a relation between \( 1/f \) and \( \Gamma \).

0: \# f(1) = \prod_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right) e^{-1/n} = \lim_{n \to \infty} \frac{1}{n} \prod_{k=1}^{n} \left( 1 + \frac{1}{k} \right) e^{-1/n}
\[ = \lim_{n \to \infty} (n+1) \cdot e^{-H_n}. \]

Taking logarithms we conclude that

\[
f_1 = -\log f(1) = \lim_{n \to \infty} H_n - \log(n+1)
\]

\[
= \lim_{n \to \infty} (H_n - \log n) + \log \frac{n}{n+1} \to 0
\]

\[
= \lim_{n \to \infty} (1 + \frac{1}{2} + \cdots + \frac{1}{n}) - \log n \quad \text{exists.}
\]

\[ f = 0.5772 \ldots \] is called the Euler–Mascheroni constant. It is not known whether \( f \) is irrational or not.

So \( f(1) = e^{-f} \).

Let

\[ g(z) = e^{f_z} \cdot f(z) = e^{f_z} \cdot z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) \cdot e^{-z/n}. \]

Then \( g(1) = e^{f_1} \cdot e^{-f} = 1 \).

\( g \) is analytic with simple zeros at \(-n, n \in \mathbb{N}\).

For \( z \in \mathbb{C} \):

\[ g(z) = e^{f_z} \cdot z \cdot \lim_{n \to \infty} \prod_{k=1}^{n} \left(1 + \frac{z}{k}\right) \cdot e^{-z/k}
\]

\[ = e^{f_z} \cdot z \cdot \lim_{n \to \infty} \prod_{k=1}^{n} \left(\frac{z+k}{k}\right) \cdot e^{-z/H_n}
\]

\[ = e^{f_z} \cdot \lim_{n \to \infty} z \frac{(z+1) \cdots (z+n)}{n!} \cdot e^{z \left(\log(1+H_n) - \frac{z}{n}\right)}
\]

\[ = \lim_{n \to \infty} z \frac{(z+1) \cdots (z+n)}{n! \cdot n^z} \cdot e^{z \left(\log(1+H_n) - \frac{z}{n}\right)}
\]

Hence for \( z \in \mathbb{C} \), \( f \) is
\[ g(z+1) = \lim_{n \to \infty} \frac{(z+1) \cdots (z+n)}{n! \cdot n^z} \]

\[ = \frac{1}{z} \lim_{n \to \infty} \frac{z (z+1) \cdots (z+n)}{n! \cdot n^z} = \frac{1}{z} g(z) . \]

Let \( \tilde{g}(z) = 1/g(z) \).

Then i) \( \tilde{g} \) is meromorphic on \( \mathbb{C} \).

ii) \( \tilde{g}(1) = g(1) = 1 \).

iii) \( \tilde{g}(z+1) = \frac{1}{z} g(z+1) = \frac{1}{z} \tilde{g}(z) = z \tilde{g}(z) \).

iv) Let \( S = \{ z \in \mathbb{C} : y \in \mathbb{R}, \Re z = z \} \).

Then for \( z = x + iy \in S, 1 \leq x \leq 2, y \in \mathbb{R}, \)

\[ |\tilde{g}(z)| = \lim_{n \to \infty} \left| \frac{n! \cdot n^z}{z \cdot (z+1) \cdots (z+n)} \right| \]

\[ \leq \lim_{n \to \infty} \frac{n! \cdot n^x}{x \cdot (x+1) \cdots (x+n)} = \tilde{g}(x) \]

Note: \( |z| = |e^{(x+iy)}| = e^{x} \cdot |\Re z| = \Re z, \]

\[ |z+k| = \Re (z+k) = x+k, \quad k = 0, \ldots, n. \]

Since \( g \) is zero-free on \([1,2]\), \( \tilde{g} = 1/g \) is odd on \([1,2]\).

Hence

\[ \sup_{z \in S} |\tilde{g}(z)| \leq \sup_{x \in [1,2]} |\tilde{g}(x)| < \infty. \]

The properties i) iv) characterize the \( \Gamma \)-function unique (246 B1 HW 9, Prob. 3).

Hence \( \tilde{g} = \Gamma \).

Conclusion \( 1/\Gamma \) is an entire function.

and

\[ \frac{1}{\Gamma(z)} = e^{\frac{z}{1}} \cdot \frac{\prod_{n=1}^{\infty} (1 + \frac{z}{n})}{e^{-2}} \cdot e^{\frac{z}{n}} \]

\[ = \lim_{n \to \infty} \frac{z \cdot (z+1) \cdots (z+n)}{n! \cdot n^z} . \]