1. This follows by checking the definition of an automorphism for $\alpha \circ \beta$ and $\alpha^{-1}$. I’ll just do it for $\alpha \circ \beta$. Clearly $\alpha \circ \beta$ is a bijection $A \to A$, since the composition of two bijections is a bijection. Moreover, if $c$ is a constant symbol, then $(\alpha \circ \beta)(c) = \alpha(\beta(c)) = \alpha(c) = c$

since $c = \alpha(c) = \beta(c)$. If $f$ is an $n$-place function symbol and $a_1, \ldots, a_n \in A$, then

$(\alpha \circ \beta)(f(a_1, \ldots, a_n)) = \alpha(\beta(f(a_1, \ldots, a_n)))$

$= \alpha(f(\beta(a_1), \ldots, \beta(a_n)))$

$= f(\alpha(\beta(a_1)), \ldots, \alpha(\beta(a_n)))$

$= f((\alpha \circ \beta)(a_1), \ldots, (\alpha \circ \beta)(a_n))$.

Here in the second equation we used that $\beta$ is an automorphism, and in the third equation we used that $\alpha$ is an automorphism. For an $n$-place relation symbol $R$ and $a_1, \ldots, a_n \in A$ we have

$(a_1, \ldots, a_n) \in R \iff (\beta(a_1), \ldots, \beta(a_n)) \in R$

since $\beta$ is an automorphism of $\mathfrak{A}$. Since $\alpha$ is an automorphism of $\mathfrak{A}$:

$(\beta(a_1), \ldots, \beta(a_n)) \in R \iff (\alpha(\beta(a_1)), \ldots, \alpha(\beta(a_n))) \in R$.

Hence

$(a_1, \ldots, a_n) \in R \iff ((\alpha \circ \beta)(a_1), \ldots, (\alpha \circ \beta)(a_n)) \in R$.

This shows that $\alpha \circ \beta$ is an automorphism of $\mathfrak{A}$.

2. The automorphisms of $\mathfrak{F} = (\mathbb{Z}, <^3)$ are exactly the maps $x \mapsto x+k: \mathbb{Z} \to \mathbb{Z}$ (for a constant $k \in \mathbb{Z}$). (So the map $\alpha \mapsto \alpha(0)$ is an automorphism of the automorphism group of $\mathfrak{F}$ onto the group $(\mathbb{Z}, +)$.)

3. Clearly $\emptyset$ is defined by the formula $\neg v_1 = v_1$. Suppose that $\varphi$ and $\psi$ with $\text{fr}(\varphi), \text{fr}(\psi) \subseteq \{v_1, \ldots, v_k\}$ define $D$ and $E$, respectively, that is,

$D = \{(a_1, \ldots, a_k) \in A^k : \mathfrak{A} \models \varphi[a_1, \ldots, a_k]\}$

and

$E = \{(a_1, \ldots, a_k) \in A^k : \mathfrak{A} \models \psi[a_1, \ldots, a_k]\}$. 
(a) We have
\[ D \cap E = \{ (a_1, \ldots, a_k) \in A^k : \mathfrak{A} \models (\varphi \land \psi)[a_1, \ldots, a_k] \}, \]
\[ D \cup E = \{ (a_1, \ldots, a_k) \in A^k : \mathfrak{A} \models (\varphi \lor \psi)[a_1, \ldots, a_k] \}, \]
\[ A^k \setminus D = \{ (a_1, \ldots, a_k) \in A^k : \mathfrak{A} \models \neg \varphi[a_1, \ldots, a_k] \}, \]
showing that \( D \cap E, D \cup E \) and \( A^k \setminus D \) are definable in \( \mathfrak{A} \).

(b) We have
\[ \pi(D) = \{ \pi(a_1, \ldots, a_k) : (a_1, \ldots, a_k) \in D \} \]
\[ = \{ (a_1, \ldots, a_{k-1}) \in A^{k-1} : (a_1, \ldots, a_{k-1}, a_k) \in D \text{ for some } a_k \in A \} \]
and hence
\[ \pi(D) = \{ (a_1, \ldots, a_{k-1}) \in A^{k-1} : \mathfrak{A} \models \exists v_k \varphi[a_1, \ldots, a_{k-1}] \}. \]
This shows that \( \pi(D) \) is definable in \( \mathfrak{A} \).

4. (a) Here is an inductive definition of the set of positive formulas:
- Every atomic formula is positive;
- if \( \varphi, \psi \) are positive, then \( (\varphi \rightarrow \psi) \) is positive;
- if \( \varphi \) is positive, then \( \forall v_i \varphi \) is positive.

(b) Consider the \( S \)-structure \( \mathfrak{A} \) whose universe consists of a single element \( a \) (so \( A^n = \{(a, a, \ldots, a)\} \) for every \( n > 0 \)), and where
- every \( n \)-place function symbol \( f \) is interpreted as the function \( (a, a, \ldots, a) \mapsto a : A^n \to A \);
- every \( n \)-place relation symbol \( R \) is interpreted as the relation \( R^\mathfrak{A} = \{(a, a, \ldots, a)\} \);
- every constant symbol \( c \) is interpreted as \( c^\mathfrak{A} = a \).

There is only one assignment \( s \) in \( \mathfrak{A} \), and clearly every atomic formula holds in \( \mathfrak{A} \) with \( s \). By induction on the construction of positive formulas, it follows that \( \mathfrak{A} \models \varphi[s] \) for every positive formula \( \varphi \). (I am leaving out some details here which you were supposed to provide!)

5. Among many possible solutions, here is one: Let \( \varphi \) be the sentence
\[ \forall x \forall y (fx = fy \rightarrow x = y) \land \exists z \forall x (\neg fx = z). \]
A structure \( \mathfrak{A} = (A, f^\mathfrak{A}) \) satisfies \( \varphi \) exactly if the map \( f^\mathfrak{A} : A \to A \) is injective, but not surjective. But a set \( A \neq \emptyset \) is infinite if and only if there exists a map \( A \to A \) which is injective and not surjective. Hence \( \varphi \) cannot hold in \( \mathfrak{A} \) with finite universe \( A \). For a language with a single 2-place relation symbol \( R \), let \( \varphi \) be a sentence which expresses that \( R \) is an ordering without right endpoint:
\[ \varphi = \forall x Rxx \land \forall x \forall y \forall z (Rxy \land Ryz \rightarrow Rxz) \land \forall x \exists y Rxy \]
Any structure \( \mathfrak{A} = (A, R^\mathfrak{A}) \) satisfying \( \varphi \) has infinitely many elements.