Problem Set 3
Solutions

Mathematical Logic
Math 114L, Spring Quarter 2008

1. (a) We proceed by induction on \( n \) to show that given a set \( \Sigma \) consisting of \( n \) wffs there exists an independent equivalent subset \( \Sigma_0 \) of \( \Sigma \). If \( n = 0 \), then there is nothing to show, since \( \Sigma \) is then automatically independent. Suppose \( n > 0 \). If \( \Sigma \) is already independent, we are done. If not, let \( \alpha \in \Sigma \) with \( \Sigma' := \Sigma \setminus \{\alpha\} \models \alpha \). Then clearly \( \Sigma \) and \( \Sigma' \) are equivalent: if \( \Sigma' \models \beta \) then \( \Sigma \models \beta \) since \( \Sigma' \subseteq \Sigma \); and if \( \Sigma \models \beta \), and \( v \) is a truth assignment satisfying \( \Sigma' \), then \( \bar{v}(\alpha) = T \) since \( \Sigma' \models \alpha \), hence \( v \) satisfies \( \Sigma = \Sigma' \cup \{\alpha\} \) and thus also \( \beta \), so \( \Sigma' \models \beta \). Since \( \Sigma' \) has \( n - 1 \) elements, by inductive hypothesis there exists an equivalent independent subset \( \Sigma'_0 \) of \( \Sigma' \). Then \( \Sigma \) and \( \Sigma'_0 \) are also equivalent. (So we may take \( \Sigma_0 := \Sigma'_0 \).)

(b) Consider \( \Sigma = \{A_1, A_1 \land A_2, A_1 \land A_2 \land A_3, \ldots, A_1 \land \cdots \land A_n, \ldots\} \).

(c) The equivalent independent subsets are \( \{\alpha \land \beta, \beta \land \gamma\} \) and \( \{\alpha \land \beta \land \gamma\} \).

2. Take \( \alpha = (A_1 \land A_1) \), \( \beta = A_1 \). Then \( (\alpha \land \beta) = (\gamma \land \delta) \) where \( \gamma = (A_1 \land A_1) \) and \( \delta = A_1 \) with \( \alpha \neq \gamma \).

3. Let \( v \) be the truth assignment with \( v(A_n) = T \) for all \( n \). We claim that \( \bar{v}(\alpha) = T \) for every positive wff \( \alpha \). We show this by using the induction principle. If \( \alpha \) is a sentence symbol, then the claim holds trivially: \( \bar{v}(\alpha) = v(A_n) = T \). Otherwise \( \alpha = (\beta \land \gamma) \) where \( \beta \) and \( \gamma \) are positive wffs and \( \square \in \{\land, \lor\} \). By inductive hypothesis we have \( \bar{v}(\beta) = \bar{v}(\gamma) = T \); hence also \( \bar{v}(\alpha) = T \).

4. Can be done in a similar way as the Example on p. 50 of the textbook.

5. Consider the set \( \Sigma_n \) consisting of all wffs of the form \( \square_1 A_1 \lor \cdots \lor \square_n A_n \) where each \( \square_i \) is either empty or equals \( \neg \). So we have

\[ \Sigma_1 = \{A_1, \neg A_1\}, \Sigma_2 = \{A_1 \lor A_2, A_1 \lor \neg A_2, \neg A_1 \lor A_2, \neg A_1 \lor \neg A_2\}, \text{etc.} \]

Then every subset of size at most \( n \) of \( \Sigma_n \) is satisfiable; we prove this by induction on \( n \), the case \( n = 1 \) being trivial. Suppose \( \Sigma \) is a subset of \( \Sigma_n \) of size at most \( n \), where \( n > 1 \). If every wff in \( \Sigma \) has the form \( \square A_n \) or every wff in \( \Sigma \) has the form \( \square A_n \lor \neg A_n \) then we are done: any truth assignment \( v \) with \( v(A_n) = T \) (resp. \( v(A_n) = F \)) satisfies \( \Sigma \). So suppose otherwise; so there exists a wff \( \square A_n \) and a wff \( \square A_n \lor \neg A_n \) in \( \Sigma \). Let \( \Sigma' \) be the set of all wffs \( \alpha \) such that \( \alpha \lor \neg A_n \in \Sigma \). Then \( \Sigma' \) is a subset of \( \Sigma_{n-1} \) of size at most \( n - 1 \), so by inductive hypothesis there is a truth assignment \( v' \) satisfying \( \Sigma' \). Then \( v \) defined by \( v(A_i) = v'(A_i) \) for \( i \neq n \) and \( v(A_n) = T \) satisfies \( \Sigma \).