Analysis Qual: Key Ideas and Problems To-Do
Will Feldman, Brent Nelson, Nick Cook, Alan Mackey
(Now with more Folland Problems!)
(Red is To-Do)

Spring 1970:
R2. f Lebesgue measurable and continuous at 1. g is in L^1. Evaluate the limit as n goes to
infinity of the integral from -n to n of f(1+x/n^2)g(x)dx.
Pull the indicator function 1_-[-n,n] inside and note that for large enough n the integrand is
dominated by 2f(1)g(x).
R3. Find the extreme points of the unit ball in L^2([0,1]).
That is, find points h in the closed ball, such that h=-.5(f+g) for f, g in the closed ball implies
h=f=g.
Write out the norm of h=.5(f+g) and Cauchy-Schwarz, then use ||f||,||g||<=1 to bound ||h||<=1.
Then look back at conditions for equality in those inequalities, and they imply f=g=h.

Fall 2001:
R1. (a) 1,-1,0,0,0,... new line 0, 1, -1, 0... etc. (b) Tonelli
R2. Closed and bounded. b) determinant continuous and maps to {+1,-1}
R4. 245B TA Section (density argument)
R5. Consider number of faces, vertices and edges of unit ‘ball’ polytopes
R6. 245b homework (linear functional mapping sequences to their limit)
C3. (b) induction
C4. See Sp02.11
C5. (a) uniform continuity
(b) Sequence of coefficients is square summable: use orthogonality to write L2 norm of
F(theta,r) for r fixed as sum |a_n|^2r^2n use mct as r goes up to 1 and continuity of F on the
closed disk.

Winter 2002:
1. Consider a sequence of “triangle” functions: f_n is zero on [1/n,1] and from [0,1/n] is a triangle
of height 1 with peak at 1/2n. Then f_n converges to zero everywhere but not uniformly.
[Counterexampled Prop 1.15], show induced measure is abs continuous w.r.t to Lebesgue
measure and a) (Not sure this is the right number)
3. Apply Fubini to LHS. Must verify integrand is integrable using Tonelli.
4. Define a premeasure by mu( (a,b))=F(b)-F(a) (Follanpy Radon-Nikodym
5. 245b discussion: pf by contradiction, consider preimages of diadic annuli, infinitely many
of which must have positive measure. Bound ||fg||_2 below by divergent sum for a carefully
chosen g. (can also use operator stuff, Closed Graph theorem)
6. Use a denisty argument
7. First consider X=[0,\infty). Since X has both sets of arbitrarily large and arbitrarily small
measure, given any p*, L^p* is not contained in L^q for any q>p*, and is not contained in any L^r
for any r<p*.
So that means we have f_n in L^p* such that f_n is not in L^p^{*+1/n} So we can make an f as
follows: partition the kth unit interval into \([0,1/2), [1/2,1/4),...,\) and on each let \(f\) be \(2^{k-1}f_1, 2^{k-2}f_2,\ldots\) as it is on its kth interval of length \(2^k\). This function is now in \(L^{p^*}\) but not in \(L^p\) for any \(p>p^*.\) (This construction is confusing)

We can define \(g_n\) and \(g\) in the same way to get a function in \(L^{p^*}\) but not in \(L^p\) for any \(p<p^*\).

Now define \(h\) on \([0,\infty)\) by alternating \(f\) and \(g\) on each interval.

Moving to \(R^n\), do what you have to to make a radially symmetric function \(H\) out of our \(h.\)

8. Was on 245b final.

11. Assume one is compactly supported. Then DCT gives you a power series for the other, implying it’s analytic on the complex plane.

**Spring 2002:**

5. Folland pg 187 Exercise 8

6. Not sure why measure-preserving

7. \(\text{sgn}(g)\) and \(|g|\) are in the Hilbert space

(a) Clear

(b) Assume \(<g,p>=0\) for all polynomials \(p\). First note that \(\text{sgn}(g)\) is in the Hilbert space:

\[
<\text{sgn}(g),\text{sgn}(g)> = \int_{-\infty}^{\infty} \text{sgn}(g(x))\text{sgn}(g(x))e^{-x^2}dx = \int_{-\infty}^{\infty} e^{-x^2}dx < \infty.
\]

Consequently, \(G(x):=g(x)e^{-x^2} \in L^1(R);\)

\[
\int_{-\infty}^{\infty} |G(x)|dx = \int_{-\infty}^{\infty} g(x)e^{-x^2}\text{sgn}(g(x))dx = <g,\text{sgn}(g)> \leq ||g|| \cdot ||\text{sgn}(g)|| < \infty.
\]

**OR,** you can Cauchy-Schwarz

\[
\int_{-\infty}^{\infty} |G(x)|dx = \int_{-\infty}^{\infty} |g(x)|e^{-x^2}dx = \int_{-\infty}^{\infty} |g(x)|e^{-x^2/2}e^{-x^2/2}dx
\]

this:

\[
\leq (\int_{-\infty}^{\infty} |g(x)|^2e^{-x^2}dx)^{1/2} (\int_{-\infty}^{\infty} e^{-x^2}dx)^{1/2} < \infty
\]

since \(g \in H.\)

I next claim that \(e^{x\xi}G(x) \in L^1(R);\) Indeed:

\[
\int_{-\infty}^{\infty} e^{x\xi} |G(x)|dx = \int_{-\infty}^{\infty} |g(x)|e^{-x^2+x|\xi|}dx
\]

\[
= \int_{-\infty}^{\infty} |g(x)|e^{-x^2/2}e^{-x^2/2+|x|\xi}dx
\]

\[
\leq (\int_{-\infty}^{\infty} |g(x)|^2e^{-x^2}dx)^{1/2} (\int_{-\infty}^{\infty} e^{-x^2+2|x|\xi}dx)
\]

This last quantity is finite because the first factor is merely \(||g(x)||\), and the second factor is a multiple of a translated Guassian (complete the square). Thus by the dominated convergence theorem and part (a) we have
\[ \tilde{G}(\xi) = \int e^{i\xi} G(x) dx = \int \sum_{n=0}^{\infty} \frac{i^n x^n \xi^n}{n!} G(x) dx = \sum_{n=0}^{\infty} \frac{i^n x^n \xi^n}{n!} G(x) dx = \sum_{n=0}^{\infty} \frac{i^n x^n \xi^n}{n!} G(x) dx = \frac{g(x)(i^n x^n \xi^n)}{n!} = 0 \]

Consequently, \( G(x) = 0 \) almost everywhere and therefore \( g(x) = 0 \) almost everywhere.

8. Suppose \( Uf = \lambda f \), then \( \|f\|_2 = \|Uf\|_2 = |\lambda| \|f\|_2 \) by the translation invariance of Lebesque measure. So if \( f \) is non-zero then \( |\lambda| = 1 \). Consequently, \( |f| \) is periodic with period 1, contradicting \( |f| \) being integrable unless \( f \) is identically zero.

9. \( e^{a(pi z)} \) [pay attention!] followed by \( z^{1/2} \), upper half to disk. you actually get a 1 (real) parameter family of maps this could be.

10. (a) Riemann Mapping Theorem, OR constructive: the inverse of \( z+1/z \) (recall this ex from 246a section, Huiyi)

(b) Morera

11. Holder inequality and mean value property of harmonic functions

13. Let \( B(0,r) \) be a subset of \( U \) on which \( |f'(z)| < (1-\delta) \), which can be done since \( |f'(0)| < 1 \) and \( f \) is continuous. Then by the fundamental theorem of calculus, if \( |z| < r \) then

\[ |f(z)| = |\int_0^z f'(z) dz| < |z|(1-\delta) < r(1-\delta) \]

Hence \( f(B(0,r)) \subseteq B(0,r(1-\delta)) \). Iterating this we see that \( f^{(n)}(B(0,r)) \subseteq B(0,r(1-\delta)^n) \), ergo \( f^{(n)} \to 0 \) uniformly on \( B(0,r) \). Now, since \( U \) is bounded (it is a domain) we know the \( f^{(n)} \) are a normal family, and since any uniformly convergent subsequence must have a limit which is identically zero on \( B(0,r) \), we know the limit is zero on all of \( U \). Thus every subsequence has a further subsequence which converges uniformly on compact set to zero and therefore the sequence itself converges to zero.

14. Monodromy Theorem (on 246B/C Homework)

OR: approximate \( f \) by smooth functions, then define a log for those in the usual way?

**Fall 2002:**

2. \( f^* \) is “inverse” of distribution function (Folland exercise 40 on page 199 outlines this problem nicely)

3. one direction reverse triangle, other use generalized LDCT

4. (a) density arg, OR, how about showing it with sequences and DCT?

(b) Minkowski integral inequality, OR, how about showing it’s in \( L^1 \) using Tonelli, and it’s also in \( L^\infty \), so it’s in everything inbetween? BRILLIANT!

5. Meyerson has the solution, and it’s not pretty.

6. Poincare inequality (use cube)

8. 245B homework

9. a) fubini

b) bounded by part (a) with Parseval
compact: can use arzela ascoli, or show it's the operator norm limit of the finite rank operators of projecting onto the first N fourier modes.
c) Take the fourier transform of $Tf=\lambda f$, and use the coefficients you got in part (a) to get an expression for the coefficients of $f$ (algebra). This expression implies $f(0)$ not equal to zero. But $Tf=\lambda f$ implies $f$ is absolutely continuous and $f(0)=0$.
OR: write out $Tf=\lambda f$ and you have an integral equation. If you could differentiate this equation, you'd get an ode for $f$ whose solution is $f(x)= f(0)\exp(\lambda x)$, and $f(0)=0$ implies $f=0$. Why could we differentiate? Well $f$ in $L^2$ => $f$ in $L^1$ => $f$ in $C_{abs}$ => $f$ in $C^1$
d) For $A:=T-\lambda f$ to be invertible, want $A$ to be bounded, 1-1 and onto. Bounded by $T$ bounded. 1-1 by part c).
Onto: show $M:=range (A)$ is closed and its orthogonal complement is empty.
For the latter, note the orthogonal complement is the nullspace of the adjoint, which is empty by a very similar argument to part c.
For the former, take $(T-\lambda f)g_n -> g$. Want $g=Af$ for some $f$. We would be done if we could conclude $f_n$ is a Cauchy sequence from the fact $Af_n$ is a Cauchy sequence: in other words, we want an estimate like $||g||<= C||Ah||$ for some $C>0$.
Suppose there is no such $C$. That is, suppose $\inf \{||h||=1 ||(T-\lambda f)h||=0$. Then we have a sequence on the unit ball $h_n$ such that $Th_n - \lambda h_n -> 0$. $T$ compact implies we have a subsequence (which I'll call $h_n$) such that $Th_n -> f$ for some $f$ in $L^2$. Let's show this $f$ is nonzero and an eigenvector, contradicting part c.
f nonzero: $0 = \lim(Th_n - \lambda h_n) = f - \lambda h_n$. So $||f||=||\lambda h||$ which is not zero.
f an eigenvector: $T$ is continuous, so $0 = T(\lim Th_n - \lambda h_n) = Tf - \lambda f$.
10. Hint can be used, but if you know anything about infinite products (i.e. from complex analysis) then it is fairly easy.
12. Schwarz reflection across circular boundary, then you have an analytic function that is zero on a set with an accumulation point.
13. Harnack+clopen argument
14. Set the integral of $\exp(iz^2)$ around a 1/8th pie slice contour equal to zero, rearrange. Showing the radial contour integral goes to zero is slightly less than trivial for a change.
15. See Sp02.11

Winter 2003:
1. Finite unions of dyadic squares are dense in open sets which are dense in measurable sets (since the measure is regular)
2. C-2
3. Schur's Test
4. If $h(x)$ is the Cantor function then $g(x)=h(x)+x$ is stricly increasing, bijective from $[0,1]$ to $[0,2]$ and maps the Cantor set to a set $E$ of measure one. This set has a subset which is not Lebsegue measurable, but it's preimage $A:=g^\ast{-1}\{E\}$ is a subset of the Cantor set and therefore Lebesgue measurable. Take $f=g^\ast{-1}$.
5. Indicator functions on subsets of the natural numbers, or use the “R is uncountable” diagonal trick
6. Show $||-||$ is continuous with respect to $||-||^*$ and then use the fact that the unit ball of $||-||^*$ is compact to show that $||-||$ achieves its min and max there. Or say “open mapping theorem ->
bounded inverse”.  
7. Use induction: zero is a closed subset then prove (using Hahn-Banach) that if M is closed, M+Cx is closed (C being the complex numbers).  
8. Use Fourier transform  
9. \((z+i)/(z-i), \ldots\) arcs become arcs of circles connecting 1 and -1  
10. at zero they converge to infinity. then the set on which they converge to infinity is open and closed and nonempty? show uniformly on \(\text{cpt subsets}\)?  
11. Sub \(z=e^{|i\theta|}\). For the second part, as long as \(z\) doesn’t cause the function to have a zero on the boundary of the unit disc the integral should exist.  
12. [Initial proof was incorrect because we operated under the assumption that zero was an isolated singularity] Each point in \(S\{0\}\) is an isolated singularity. Case 1: There are infinitely many essential singularities in \(S\), then we are done immediately by Casarati-Weierstrass.  
Case 2: There are finitely many essential singularities, but infinitely many poles. Assume the latter possible conclusion is false and that for some point \(w\) we have \(|f(z)-w|\) epsilon in a neighborhood of zero. Let \(g(z)=1/(f(z)-w)\), then \(g(z)\) is bounded in this neighborhood and therefore has a removable singularity at zero. Note that each pole of \(f(z)\) is a zero of \(g(z)\) and therefore we must have \(g(0)=0\), but then \(g(z)\) is identically zero, a contradiction. Case 3: There are finitely many essential singularities, and poles. In this case we can assert that zero is in fact an isolated singularity and then we either have that zero is an essential singularity, in which case we get the latter conclusion, or \(f\) is meromorphic in a neighborhood of zero.

Fall 2003:  
1. Dominated Convergence. Or Fubini?  
2. (b) Show measure is regular to obtain density of continuous functions, then Stone-Weierstrass gives density of polynomials, use this to show \(\langle f, f \rangle = 0\)  
3. Done in 245B lecture  
5. (a) Done in 245B lecture (b) Orthonormal basis converges weakly to zero (c) square and expand  
7. Comes down to approximating indicators of intervals, and remember to use \(e_0(x)\)  
8. See Sp02.10  
9. Map to slit disc, then standard steps  
11. Rouche’s Theorem  
12. (a) \(z^n\) are orthogonal (b) Similar to a problem Jack did in 246A discussion (uses the fact that if a collection of analytic functions are uniformly bounded on compact subsets then the collection is a normal family) Uniform boundedness on a closed disk \(D_r\) comes from the mean value property followed by Cauchy Schwarz trick.

Winter 2004:  
A1. a) monotonicity, b) \(\{f<a\}\) are open. *Is this the definition of USC? Can also be done with the limsup \(y->x \phi(y) <= \phi(x)\) definition, but perhaps too many quantifiers to be recorded here. Ok fine. Fix \(x_0\). Let \(\eta\).  
Then there exists \(\delta_0>0 \) s.t \(\phi(x_0) >= \sup_B(x_0,\delta) f(y) - \eta\).  
Now for \(x\) in \(B(x_0,\delta)\), there exists \(B(x, \delta)\) contained in \(B(x_0,\delta)\).  
So \(\phi(x) <= \sup_B(x,\delta) f(y) <= \sup_B(x_0,\delta) f(y) <= \phi(x_0) + \eta\).
A2. standard banach fixed pt theorem (cauchy sequence)
A3. fubini + measurability argument (composition of measurable fcns)
A4. One direction is easy (using the basic bound of \(|f|_{\infty}\) for one factor of \(|f|_{n+1}\), for the other direction use the fact that the slope of \(x^n\) gets arbitrarily large as \(n\) increases. Better way: compare alpha to the \(L^p\) norm of \(f\) (and \(L^{p+1}\) by holder) and use the proof from our homework which showed limit of \(L^p\) norms is the \(L^{\infty}\) norm.
Just realized that this follows from the problem we did in 245B and the fact that the ration test and nth root test are compatible.
A5. First write the measure as a countable combination of dirac masses, and use this to reexpress \(|F(t)|^2\) as a double sum. Now the identity will drop out nicely if we can move both the integral and the limit under this double sum. The first can be done with Fubini and the second with DCT, neither of which is hard to justify.
A6. easy
C1. cauchy riemann eqns (this is not necessarily true when \(f'(z)=0\); look at \(z^2\))
C2. First note
\[
|f(z)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z + \rho e^{i\theta})| d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z)| d\theta
\]
Now suppose equality holds for some \(z\) and some radius \(\rho\). Since we are dealing with \(|f|\) we can rotate \(f\) so that \(f(z)\) is real and non-negative. We have
\[
\frac{1}{2\pi} \int_0^{2\pi} |f(z + \rho e^{i\theta})| d\theta = |f(z)| = f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + \rho e^{i\theta}) d\theta
\]
\[
= \text{Re} \left[ \frac{1}{2\pi} \int_0^{2\pi} f(z + \rho e^{i\theta}) d\theta \right] = \frac{1}{2\pi} \int_0^{2\pi} \text{Re} \left[ f(z + \rho e^{i\theta}) \right] d\theta
\]
However, \(\text{Re}[f] \leq |f|\), so the equality of the integrals implies
\[
|f(z + \rho e^{i\theta})| = \text{Re} \left[ f(z + \rho e^{i\theta}) \right], \text{ ergo } f = \text{ real valued on this circle. Mapping this smaller disc until the upper half plane and using the schwars reflection to get a bounded entire map (f is continuous on the closure of this disc since it is a subset of D and so f's range is bounded). (Equality iff argument of integrand is constant, implying f isn't analytic.)}
C3. open mapping theorem + image of connected set is connected->conformal mapping->
Louville
C4. Squaring sends the curve to the vertical line \(x=1\), then proceed to map this half-plane to the disc and use the poisson integral formula to obtain a harmonic function on the disc whose value is zero on half of the disc and one on the other half. Or if you're working in the upper half plane you should get an arctangent (recall HW problem)
C5. Instinct is to extend integral to -infinity, but function isn't even. Instead do a keyhole path so you get a radial part which goes to zero and twice the integral you're after.

Fall 2004: did Tao write this one?
2. 245C Homework
3. a) density, b) \(f(0) > 0\), so \(f>0\) on a small interval
4. 245A homework, notes 5 problem 9.
5. Show by hand that it is injective and surjective, then use the open mapping theorem
6. Define your limit set to be \(\{x \in X \mid \text{there exists a sequence with } x_{n,j} \text{ in } K_{n,j} \text{ which converges to } x\}\)
7.
9. Rouche will only tell you there are four zeros in the unit circle. To get any more you will need
to see where the three segments of the boundary are mapped (and count how many times they
wrap around zero). Turns out for large R, a radius R quarter-circle wraps around twice. For R=1
it wraps around once. To see this, a dog-walking type argument is necessary (and this can be
viewed as intuition behind Rouche): if |g|<|f|, then argf - \pi/2 < arg(f+g) < argf + \pi/2, so that the
change in argument of f+g going along a contour must be within \pi of the change in argument
of f (which is helpful when you have narrowed the change in argument for f+g down to be some
multiple of 2\pi).

10. z-i/z+i sends to slit disc (with slit from 0 to 1) and then proceed in standard way
11. Convert from question about continuity of a function to one of convergence of sequences
of functions via g_n(z)=f(z/n). Now this is a normal family and so has a normally convergent
subsequence. The pointwise convergence to zero along the real axis implies the limit is
identically zero. Now we will in fact need the whole sequence to go to zero, and this follows
from the fact that any subsequence is a normal family that has a further subsequence that
must converge to zero by the same reasoning as above. So g_n(z) goes to zero uniformly on
compact subsets of the right half plane. Now you can consider a certain compact set such as
K={re^{i|\theta|} : .5<r<=1, -\pi<=\theta<\pi}, and you can see that the union of K/n, n=1,2,...

Fall 2005:
R1. a) You can show it is the a.e. limit of simple functions. Take a two sequences of simple
functions one whose integrals converge to the upper riemann sum one to the lower. These
sequences have limits G and g say with g<=f<=G also by MCT int(G-g)=0 so G=f=g a.e.
b) take a subset of the cantor set which is not Borel measurable its indicator fcn is riemann
integrable since its set of discontinuities has measure 0.
R2. Hahn decomposition thm
R3. previous qual
R4. a) take a basis, b) think of vector spaces which are not complete under their norm (for
example continuous functions on [0,1] with L1 norm-- T is evaluation at 1/2).
R5. a) any unbounded L1 function is not in the image of L1 under the FT. In fact DCT quickly
gives that the image of an L1 function must be continuous, and not all L1 fns are continuous.
the image of 1_{[-\pi,\pi]} is something like sin(alpha)/alpha not in L1.
In fact the FT of the indicator function of any interval is not in L1 I believe?
b) hint (haven’t done yet) (hint is wrong even?) yeah this hint is confusing me (in any case the
thing they want you to prove is off by a factor or \sqrt{2\pi}).
C1. open mapping thm: should know how to prove this from the argument principle, but note
that Rouche’s theorem nicely packages much of the argument. (elaborate please?) see Stein
Shakarchi
C2. map to slit disc then do standard stuff
C3. partial fractions, then can write 1/(z-a) and 1/(z-1/a) as power series valid in a nbhd of circle.
C4. integrating along negative axis, log(z) becomes log(-x)=i\pi+log(x), so you have to also find
the residue at i for 1/(1+x^2)^2 (I think you can avoid calculating this since that integral will be
imaginary and since the integral we are after is real it must cancel with whatever the imaginary part of $2^*pi^*i \text{Res}(f(z),i)$ is).

C5. It must mean finite Borel measure otherwise wtf. Then it is pretty straightforward.
Ways to show analytic: produce a power series (use DCT), Morera (with Fubini) and DCT to show continuous, show it’s differentiable (with DCT).
For power series at infinity you can expand integrand in a power series and switcheroo with DCT again.

Spring 2006:
1. Done in 245B section
2. Done in 245B section
3. Done in 245B section
4. Standard proof of Arzela-Ascoli
5. Holder inequality and density of continuous, compactly supported functions
6. (1) Cauchy-Schwarz and the fact that $1/n$ is square summable (2) look what you just did in (1) [compute $L(f)$ and you’ll see what $g$ has to be]. Requires Fubini-Tonelli
7. Harnack
8. Integrate over 45-45-90 triangle in the first quadrant (so that the hypotenuse satisfies $x=y$) Or a 1/8th pie slice. Proceeds much as #14Fall02. Also note that this is perhaps the only problem where it is less than trivial to show the radial part of the contour integral goes to zero as R goes to infinity, since the integrand isn’t uniformly bounded (it actually goes to infinity for theta=0), but you can either introduce a quantifier and break off epsilon/R of the contour near theta=0, or note that sin(2$\theta$)=<? for small enough theta, and you can spot an antiderivative for that one.
9. Note that if f hits zero, that is f(z) =0 for some z, then f reflects across the real axis to have a pole at $\bar{z}$, which contradicts f mapping the plane into the plane. So f not hitting zero is key for claiming there’s a well defined analytic logarithm. C is simply connected so you can define logf in the usual unambiguous way as the primitive of $f/f$.
10. Show the functions have the same poles and so their ratio (difference?) is an entire function. 246B (need to first argue the sum converges uniformly on compact sets in $\mathbb{C}\setminus \mathbb{Z}$. Boundedness follows from analytic and periodic).
11. Let h be what’s inside the log. You define log(h) on each domain as the integral from 0 to $z$ of $h'/h$. Then you note that if they are asking for $f(4\pi i)-g(4\pi i)$, this is actually a way of writing the integral of $h'/h$ around the closed contour going out along U and coming back through O. So you find the residues of $h'/h$ inside this seashell thing. There are like 5 (at 1,-1,-i/2,-i/2, 3i/2 and -3i/2). The residues there are -1, -1, +1, +1, and +1. So $f(4\pi i)-g(4\pi i) = 2\langle \pi i \rangle(4-2) = 4\langle \pi i \rangle$
(Use the argument principle so that $f(4\pi i)-g(4\pi i)=2\pi i (Z-P)$ where Z is the number of zeros enclosed in this contour (there are 4: +/- pi/2, 3pi/2, -5pi/2) and P is the number of poles (there are 2: +/- 1) so the answer should be 4pi*i)
Yes actually most of the computation in blue is hidden in the proof of the argument principle, so definitely use argument principle (esp since you might flip a sign like I did). I’ll leave it in case you want to recall why argument principle works :)
Just saw Willie Meyerson has a solution for this one, and dubs it the Hardest Complex Analysis question. It looks really long though. Is any of the extra stuff necessary? I think we have a more direct path that’s still correct.
12. Induction (binary) show it has the same series as $(1-z)^{-1}$
Fall 2006:
1. Done in 245B discussion section
2(1) See previous quals (2) Compute the convolution and then realize the result follows by 
Lebesgue differentiation theorem.
3. 245B Homework
4. 245B HW: If \( \hat{f}(n) \) is the nth fourier coefficient of f then use integration by parts to show \( |\hat{f}(n)| = |\hat{F}(n)|/n \). Then use Cauchy-Schwarz and Plancheral 
5. (1) Apply Cauchy-Schwarz twice (2) Define T on finite linear combinations of the \( e_n \) by the 
rule \( T(e_n) = e_1 + ... + e_n \) (hence it’s matrix will be upper triangular with all entries at and above the 
diagonal equal to one). Produce some extra basis vectors so that every element in the space is 
a FINITE linear combination of the original basis vectors and the new ones, and define T to be 
zero on all the new basis vectors. Then T is unbounded.
6.(1) integrate in polar coordinates and use orthogonality of exp(inx). (2) no, consider zbar for example 
7. show that f has a pole at infinity (since otherwise the image of nbhd of infinity would be 
dense in the plane by casorati weierstrass can use to contradict univalence via open mapping 
theorem) so f extends to be meromorphic on the sphere so it is rational and its only pole is at 
infy so it is a polynomial then use fund thm of algebra BAM.
WOAHHHHHHHHHHH!!
8. Compose with \( z^{-1}/z+1 \) and then apply the Schwarz lemma (and triangle inequality)
9. Contour integral. substitute \( \cos \theta = (1/2)(e^{i\theta} + e^{-i\theta}) = (z+1/z)/2 \). don’t fuck up 
10. Conformal mapping of slit disc 
11. Use the proof of the open mapping theorem involving winding numbers and use the fact that 
image of the circle \( 1+\exp(i\theta) \) under the function \( \exp(z)(z-1)^n \) stays a distance of at least 1 
away from the origin.
*OR*: apply Rouche with the region \( |z-1|=1 \). You get that \( \exp(z)(z-1)^n-\alpha \) has same number 
of roots in there as \( \exp(z)(z-1)^n \), which has a root of multiplicity n at 1. Also, the roots must be 
simple since the derivative of \( \exp(z)(z-1)^n-\alpha \) only vanishes at \( z=1 \) and \( 1-n \), which are not 
roots. To see that there are no roots outside this circle, note as above that \( \exp(z)(z-1)^n \) is larger 
than 1 out there.
12. (1) Just show that the sum of \( (a_n-z)/(1-a_nz)-1 \) converges uniformly on compact subsets. (2) 
If you show the product has a zero at each \( a_n \) then an analytic continuation would be zero since 
the zeros then have a limit point in the domain (namely 1). Also, the product has poles at \( n^2/ 
(n^2-1) \) - outside the disk and arbitrarily close to 1.

Spring 2007:
1. prove they converge in L1 norm by using fatou on \( |\hat{f}(n)| + |\hat{f}(f_n)| - |\hat{f}(f_n-f)| \) (i.e. proof of 
generalized DCT).
Or, let \( g_n = |\hat{f}(n)| - |\hat{f}(n+f)|. \) Then \( g_n \geq 0 \), converge pointwise to zero and \( g_n \leq 2f \). So by the DCT 
\( \int g_n = 0 \), which implies \( 0 \leq |\hat{f}(f_n) - 1+1. 
2. pull out an \( x^2 \) from the bottom and then think of it as a riemann sum of sorts to put it into a 
nicer form, answer we got is \( 1/\sqrt{\pi} \)
I got \( \pi / (2\sqrt{\pi}) \). You do the squeeze theorem, bounding the sum above and below by an 
integral (it’s a cauchy distribution: do \( \theta = \arctan(y) \) substitution), and these both converge to
the same thing as $x \to \infty$.

3. (1) Fubini (2) $|A_m| < |g|$ is clear, for other direction use any positive $f$ with norm one and fubini (3) Can use that Fourier Trans of conv. is multiplication, to see that $^\wedge f(x)$ is 0 or 1 for a.e $x$, but since $^\wedge f$ has a continuous version it is zero or one everywhere., but then Riemann-Lebesgue implies we must have $^\wedge f = 0$.

Or you can use easy change of measure and translation invariance to show that $f^*f$ is translation invariant, ie constant, so zero.

4. (1) let $M = \{f : Tf = f\} = \{f : T^*f = f\} = N(l-T)^*$ since T unitary so $R(l-T) = M$ orthogonal complement, so for all $f$ in L2 $f = g + (l-T)h$ (where $g$ is in $M$), suffices to show $1/n^*Sn((l-T)h) \to 0$ for all $h$ in L2. (since $1/n^*Sn(g) = g$ for $g$ in $M$)

See the last Hilbert space problem in Folland for the above.

The Nick and Alan “hard analysis” way: (goddamnit will/folland)

If you write out $|^{\wedge}((1/n)S_n f(k))|^{\wedge 2}$ (modulus squared of the Fourier coefficients of $Sn f(n)$), for $k$ nonzero you get $(1/N^{\wedge 2}) |^{\wedge}f(k)|^{\wedge 2} |G(k,N)|^{\wedge 2}$, where $G^{\wedge 2}$ is a rational function of trig functions. So if you sum this over the nonzero integers and take the limit as $N \to \infty$, you can take the $N$ limit inside the sum by the Dominated convergence theorem (counting measure, and the summands are dominated by some constant times the squared fourier coefficients of $f - G(k,N)$ is the sum of $N$ things each <1 in modulus, so $|G(k,N)| < N$). The summands converge pointwise to zero, so by Parseval’s theorem we’ve shown $S_n f/n$ converges in $L^{\wedge 2}$ to the zeroth coefficient, which is the zeroth coefficient of $f$, which is the average of $f$, which makes sense.

if this is the one with the shifts then yea this should work too, and its probably the better way to do it on a test (in general if you hadn’t seen one of these ergodic averages problems before) since when you see a shift of an L2 function you should probably think to turn in into a modulation on the fourier side.

(2) Show that the $1/n Sn(l-T)h$ are equicontinuous and apply arzela-ascoli. And here we also use the fact that any subsequence has a subsequence converging uniformly to the same limit, right? yea I’ve been using this trick a lot...

5. Hahn - Banach

6. A certainly needs to be bounded, the problem may still be wrong with the definition of $\rho(A)$ - given. Actually its true (still need to assume bddness)! Suppose $A$ is onto and $K$ is some operator with small (to be determined) norm. We want to solve $(A-K)x = u$ for $u$ arbitrary. The idea is to show that the iteration $A x_{n+1} = K x_n + u$ converges when the norm of $K$ is sufficiently small. You need to use the fact that you can solve $Az = y$ for all $y$ in the space for a $z$ with $\|z\| \leq C \|y\|$ where $C$ is independent of $y$. This is possible because we can define $A$ as an invertible bounded linear map $X / N(A) \to X$ where $N(A)$ is the nullspace of $A$ and $X / N(A)$ is a Banach space under the norm $\|y + N(A)\| = \inf x + A = y + A \| x \|$ (using the fact that $N(A)$ is closed). Now you can show the iteration converges by solving
\[ A(x_{n+1} - x_n) = K(x_n - x_{n-1}) \text{ for } \|x_{n+1} - x_n\| \leq C \|K\| \|x_n - x_{n-1}\| \text{ and then choosing } \|K\| < 1/C. \]

7. Substitute \(0.5(z+1/z)\) for \(\cos\theta\) and \(-0.5i(z-1/z)\) for \(\sin\theta\). You get three poles, two of which aren't in the unit disk. The other is at zero. I got \(-2\pi i/(1+a)\).

8. Comparing \(z^4 + 4z^2 + 3\) with \(2z + z^3\) is tempting, but the latter is actually bigger than the former at \(iy\) for some \(0 < y < 1\) (takes a while to confirm this).

So without Rouche: take a positively oriented semicircular contour of radius \(R\). For large \(R\), the portion on the \(y\) axis gets mapped to a contour that wraps almost almost 2 times clockwise around the origin. The radial part behaves like \(z^4\) for large \(R\) and thus wraps roughly \(4\pi = 2\) times counterclockwise.

So there are no zeros in the right half plane. (The part on the \(y\) axis warps around only once clockwise and the radial part wraps around once clockwise, here's the plot for \(R=2\) [any higher and you can't see the behavior near zero]:

For large \(R\) it wraps twice. \(R=2\) is not large and that's why you aren't seeing it. It's not hard to "plot" this by hand: you just look for the zeros of the real and imaginary parts of the image of the contour (\(\text{Re and Im parts of } f(iy)\)) to see when it's crossing an axis.

The answer is still zero though.

The plot shows there are no zeros inside the semicircle of radius 2, which is consistent with there being no zeros inside the right half plane.

9. First note that \(f\) can't have an essential singularity (if it did it's Laurent series at zero would have infinitely many singular terms and therefore so would it's derivatives, contradicting the
assumed bounds). Then if \( f \) had a pole of order \( n > 1 \) then \( f^{(m)} \) would have a pole of order \( n + m \), again contradicting the assumed bounds.

10. Conformally map to the disc and then use the Poisson integral formula to get the appropriate harmonic function on the disc.

11. WLOG it has no pole at infinity (multiply by sufficiently large power of \( z \)) then subtract off principal parts to obtain a function which is analytic on the sphere, ergo constant.

12. first Harnack and clopen argument get ptwise cvgce to zero. Then fixing cpt subset, Harnack on subcover gives uniform.

**Fall 2007:**

1. Show \( |^nf(n)| = |^nf(n)|/n \) and apply Parseval-Plancherel. If equality holds then \( ^nf(n) = 0 \) for all \( n > 1 \) and thus \( f(x) = c \exp\{ix\} \) for some constant \( c \).

2. Take balls of radius \( \epsilon \) around enumerated rationals, then take complement.

3. (a) Finite dimensional subspaces are closed and nowhere dense, Baire Category Theorem

(b) sequences of finite support

4. Vitali set

5. (a) alternating series test (b) split up integral into sections where cosine is alternatively positive and negative

6. Taylor series of arctan (Basic exam problem, though definitely easier now with DCT)

7. On 245B final

8. Use Stone-Weierstrass to show trig polynomials are dense.

9. (a) Let \( A \) be the integral of *dh:= -h, dx+h, dy around a circle (divided by \( 2\pi i \)). This makes the integral of *d(u_A), where \( u_A(z) = h(z) + A \log|z| \), around closed contours of a homology basis zero, so we can make a well-defined conjugate harmonic function for \( u_A \) as an integral from some point \( z_0 \) to \( z \) of *du_A, taken along any contour connecting these two points.

(b) Real way to do this problem: If \( A = 0 \) then \( h(z) \) is the real part of some holomorphic \( f \), but the \( \exp(f) \) is bounded (since \( h \) is bounded) and therefore has a removable singularity at zero. It cannot be zero since \( h \) is bounded away from negative infinity, so the real part of its log is \( h(0) \). If \( A \) is non-zero, let \( f \) be holomorphic with real part \( h(z) - A \log(z) \). Then \( z \exp(f(z)/A) = \exp(f(z)/A + \log(z)) \) is bounded since the real part of \( f(z)/A + \log(z) \) is \( h(z) \). Once again we have that the singularity at zero is removable and since the the disc is simply connected there is a homomorphic \( g \) such that \( \exp(f(z)/A + \log(z)) = \exp(g(z)) \). Hence \( h(z) = ARe(g(z)) \).

10. apply the MVP/C-S trick to the largest ball centered at \( z \) still contained in \( U \). then the numerator is the \( L^2 \) norm of the restriction to the ball, and the denominator is the square root of the area of the ball. Numerator bounded by \( L^2 \) norm over \( U \), denominator bounded below by distance from \( K \) to boundary of \( U \).

11. Use strong induction: if \( F(z) = z + a_1 z^{a_n} + \text{stuff} \), then \( F \) composed with itself \( m \) times will be \( z + ma_1 z^{a_n} + \text{other stuff} \). Realize that the compositions are uniformly bounded and therefore converge to something by arzela ascoli, so \( a_n = 0 \) otherwise the coefficient of \( z^{a_n} \) grows indefinitely.

**OR “hard” analysis version:** the \( n \)th derivative of the \( k \)th composition contradicts Cauchy
estimates for large k.
12. Standard
13. (a) Use Cauchy integral formula (b) If we had another, then take the integral of their
difference divided by $z^{n+1}$ to show that $a_n - b_n = 0$ (use strong induction first)

Spring 2008:
1. use the hint (baire category) with the open dense sets $U_n = \{ f^n (n) = 0 \}$
   What complete space are you considering? You can specialize to the space of continuous
   functions on $S^1$ with uniform topology. Can also do this one by contradiction with the countable
   union of nowhere dense sets approach.
2. worth doing (easy)
3. (a) and (b) are standard, for c) use urysohn to put cts indicator fcns on the countable base
   for topology these clearly separate points since metric spaces are hausdorff, then look at the
   algebra of polynomials with rational coefficients in these functions, this will be dense by stone
   weierstrass. (be careful to construct it such that it is still countable).
   That was nice - was stuck for a while considering set of linear combinations of the continuous
   bumps, but that’s a vector space not an algebra. So just make it an algebra! (polynomials). duhh
4. use CS for bddness then use it again along with translation is continuous in $L^p$ to get
   continuity.
5. (1) use the geometric series
   (2) show that its complement (the resolvent set) is open and contains a nbhd of infty. More
   precisely for $|z| > ||T||$ we have $||1/z^* T|| < 1$ and $(T-zI) = z(1/z^* T - I)$ so use part (1), as for the
   openness if $A$ is invertible ($A + \epsilon I = A^*(I + \epsilon A^*(I) - I)$ which is invertible for $\epsilon < 1/||A^*(I)||$).
6. the hint given in the problem works, diagonalize to find a subseq which converges on a
   countable dense, use density to show it converges for all $f$ in $C([0,1])$ and thus defines a cts
   linear functional on $C([0,1])$ then use riesz repn theorem.
7. use louivile’s theorem on $u(R(x,y)) - u(x,y)$.
8. they must have forgotten to say convex. verily.
   What if noone got up to ask and they didn’t correct it, and you just turned in the example of the
   unit circle in the plane and the origin? Would you get anything? You’d get banished.
9. contour integral. substitution $x = e^{i\theta} y$, then you get an integral over the real line. Extend to fn on
   C and you have poles at the odd half-integer multiples of $\pi$. Then you can go to the right along
   the real axis and circle back on that line shifted up by $\pi$, circling one pole at $\pi i/2$.
11. Let $g$ be the unique (up to rotation) Moebius transformation which takes $a$ to $0$, then $g f g^{-1}$
    sends $0$ to zero and fixes $g(b)$, so by the Schwarz lemma it is equal to a rotation, but the rotation
    must be by one since it fixes $g(b)$ (not just $|g(b)|$).
12. look at contour integrals around circles of $1/z$ and of its approximating polynomials
13. Easiest to use Rouche’s theorem (or just prove it using the argument principle).

Fall 2008
1. (1) Chebyshev, (2) look at $E_n, \epsilon$ sets. Or look up this part of the proof of Riesz Fischer in
   Folland (make a telescoping sum)
2. no take for example finite support functions on the natural numbers this is a countable union
   of finite dim vector spaces which all must be nowhere dense contradiction
   Or exhibit a Cauchy sequence converging to something without finite support.
3. show omega is USC the $\{ \omega(x) = 0 \}$ is the set in question and it is a countable intersection
of open sets so the set is actually G\delta? (omega being USC implies that the preimage of \(-\infty,a\) under omega is open for all a, and since these sets generate the borel sigma algebra this implies omega is measurable, whence the preimage of zero is measurable).

4. 245B HW

5. Note this problem is not correctly stated. The following assumes the measure is not supported at 0 (For a counterexample take \(\mu = \delta_0 - \delta_{2n}\)). do it for trig poly's first then use density. There is that old stumbling block of trig polys only being dense in C(T), T the torus. Start with g in C([0,2\pi]) close to f, then h in C(T) equal to g on \([\delta, 2\pi]\), taking \(\delta = \epsilon / ||g||_u\), and \(|h| = ||g||_u\). We encountered this detail in 245b homework, but there it was different as we had the advantage of “f” being essentially bounded (by 1).

6. (a)The set in question is the intersection of the complements of each “fat Cantor” set. C_n^c has measure \(\frac{1}{2^n}\) so continuity from above implies their intersection has measure zero. (b) Otherwise E would be the union of a measure zero set and a countable collection of Lebesgue measurable sets (c) One can easily extend the function from problem 4 of Winter 2003 into a function onto R c) You can build a function (How?) much like the Cantor-Lebesgue function, but strictly increasing, that sends C_n onto C_0 and C_n^c onto C_0^c (where n is st the intersection of E with C_n is not measurable).

Let C_n^k be the sets that are intersected to get C_n. Let f_k be piecewise linear, with slope (whatever_1) on C_n^k and slope (whatever_2) on the complement of C_n^k, with the whatevers chosen so that the middle \(\frac{1}{2^n}\) thirds are sent to the middle thirds in the complement of C_0^k and C_n^k is sent onto C_0^k.

How about this: f_k is the unique increasing piecewise linear function sending C_n^k onto C_0^k and the complements onto the complements....

Let f be the uniform limit of the f_k.

Now take f to be the inverse of f_n. Sends a subset of C_0 (a null set) to E’s intersection with C_n. Not sure how much effort we should put into rigorously defining this uniform limit of piecewise linear functions - it's tedious but obviously works just like the construction of the devil's staircase function.

7. (a) Show that the difference of the integrals of the holomorphic function are zero using a keyhole contour, then parameterize the integral in theta. (b) Same as problem 9 on Fall 2007 (and then use part (a))

OR with Green’s identity applied to u=logr, v=h. Get rlogM’ = M + const. Solve that ODE!

8. Consider it’s Laurent series centered at zero and show that it cannot have negative power terms if the integral is to remain bounded (use orthogonality of differing powers of z)

Or totally different: Apply the MVP/C-S trick on a ball centered at z with radius (1-\epsilon)\{|z|\}. You get \(|f| <= C||f_1B||_2 / |z|\), where that L2 norm is of f restricted to the ball that’s shrinking as z->0. But this means \(|z f(z)| -> 0 as z ->0.

9. (a) Precompose g with f(z)=(z+1)/(1-z), then gf is a conformal map from the disc to itself which fixes zero and thus is a rotation: gf=exp(i theta)z. (b) Apply Harnack’s inequality to the imaginary part of f

10. (a) Use the following: dist(K, boundary of U)>0, \(|f_n|<1 for all n, Cauchy integral formula (b) diagonalization (Why can’t you just use Arzela-Ascoli at this point?) Arzela Ascoli gives you a subsequence for a fixed compact subset. I think what “diagonalization” is referring to here is that you can get a subsequence by Arzela Ascoli on each of a sequence of domains K_n that
increase to fill the domain. Then you take the diagonal subsequence of those!
(this is to address the problem that there is a difference between “subsequence converging
uniformly on compact subsets” and “for every compact subset there is a subsequence
converging uniformly”; fortunately the latter (given by A-A) implies the former by this further
diagonalization we’re talking about).

11. First note that since $f$ is continuous on the closed unit ball, it is bounded. Change the domain
to the upper-plane, then since the imaginary part of this function is zero we can schwarz-reflect
to obtain an entire bounded function

OR

Consider $\text{im}(f)$, which is harmonic on $D$, continuous on the closure, and 0 on unit circle. By
maximum (and minimum) principle for harmonic functions, $\text{im}(f)$ is identically 0, so $f$ must be
constant by open mapping theorem.

12. usual substitution for $\sin$, then change again to $w=z^2$ to make it an integral around the
boundary of the disk. $\pi / (a \sqrt{a^2 + 1})$

Spring 2009:

1. Start work with right side, rewrite integral as double integral of a function which is 1 iff
$f(x) \leq t < g(x)$ or $g(x) \leq t < f(x)$

2. (a) 245B Homework (b) Let $\{\varphi_n\}$ be a countable, orthonormal sequence. Define

$$T_N(x) = \sum_{n=1}^{\infty} \langle x, \varphi_n \rangle \varphi_n$$

3. If $f_n$ is a countable dense set in $X$ then let $x_n$ be such that $\|x_n\|=1$ and $|f_n(x_n)|>\|f_n\|/2$, show
that if these are not dense then you can find $f$ in $X$ such that $f(x_n)=0$ for all $n$ yet $\|f\|>0$.

4. (a) Extend $f$ to the real line by letting it constantly $f(0)$ below 0 and constantly $f(1)$ above 1,
then use difference quotients (b) Use the hint (sans reducing to a subsequence) yeah what the hell

5. Use the lebesgue differentiation theorem; aw dawg the LDT!

6. (a) just do it.

(b) Show that the $h_{n,j}$’s are a complete orthonormal set in the subspace of mean zero functions
in $L^2([0,1])$, assume $f$ is orthogonal to all the $h_{n,j}$’s then show that $f$ is actually orthogonal to all
the $\chi_{I_{n,j}}$ by induction, then use the previous problem to imply $f=0$ a.e.

Note Thiele has some lecture notes on Fourier analysis (2nd quarter) that does this is the first
section.

c) The nth term is $E_n f(x)-E_{n-1} f(x)$ (check for small cases like $n=1,2$ to make the algebra more clear).
But then isn’t the $n=0$ term necessary? I think so

7. (a) show $F(z)$ is in $L^1_{\text{loc}}(C,m)$ first by using Tonelli, then use Markov or just recall that $L^1$
functions are finite almost everywhere.

(b) Apply Tonelli to $[-n,n] \times [0,1]$; it suffices to prove it for this set. The set $\text{An}$ of $y \in [0,1]$ such

$$\int |F(x+iy)| \, dx = \infty$$

has measure 0, and so $A = \cup A_n$ has measure 0. In $[0,1] \setminus A$, each
line satisfies the desired condition.
c) use Fubini now that we know our function is absolutely integrable with respect to the product
measure of \( \mu \) with the length measure on the boundary of almost every square in \( C \). But
the cauchy integral formula will give a function undefined on the boundary of \( S \). So, we need
the lebesgue measure of the set of all \((z_1,z_2)\) determined by squares which have \( \mu \)-positive
boundary measure to be 0. To prove that, first note that the set of all horizontal (or vertical)
lines with \( \mu \)-positive measure is countable (since uncountable sums of positive numbers are
infinite, and \( \mu \) is finite). Let \( E \) be the union of all horizontal and vertical lines in the plane which
are given positive measure by \( \mu \), this is a countable union of 2-dim lebesgue measure zero
sets and hence has 2-dim lebesgue measure zero. So if \((z_1,z_2)\) determines a square whose
boundary has positive \( \mu \) measure then either \( z_1 \) or \( z_2 \) must lie \( E \) i.e. \((z_1,z_2)\) is in \( \text{ExC U CxE} \)
which is a union of two 4-dim lebesgue measure zero sets, and hence has 4-dim lebesgue
measure zero.

8. Suppose for contradiction that \( f \) misses a point \( w \) then show that it must also miss \( 1-w \), then
use big picard to imply \( f \) is constant.

Is there a more elementary proof? Picard is a powerful theorem (is it fair game to use?) unless
they ask you to prove it (or it wasn’t covered in class) you can use it, its not the basic.
Okay, i’ve just heard using Riemann mapping theorem is not always okay (because it’s too
powerful). I guess the guide is to not use it if it makes the problem TOO trivial...

What problem exactly would the Riemann mapping theorem make trivial? Spring 2002 #10a
Note that this is part (a) of a problem; I’m pretty sure Riemann mapping theorem is ok for this.

9. divide out the blaschke product with the same zeros as \( f \) to get \( f/g = h \) where \( h \) is analytic and
nonzero on the ball of radius \( R \), thus \( \log(|h(z)|) \) is harmonic so use the mean value property and
the fact that \( |f(z)| = |h(z)| \) on the boundary of the ball.

(b) ? Is this the same as the 246B final problem? It seems similar, but I didn’t have such a hard
time with it then...
a solution: writing out their hint, let \( N(R) \) be the number of roots inside the ball of radius \( R \). Using
part (a) applied to a circle of radius \( 2R \) we get
\[
\log_2 N(R) \leq \log(C/|f(0)|) + (2R)^\lambda
\]
Let \( M(R) = \text{RHS} / \log_2 \). We’ll use the inequality \( N(R) \leq M(R) \) to get a sequence of radii \( R_k \n such there are at most \( k \) roots inside the circle of radius \( R_k \). That is, \( R_k \) solves \( M(R_k) = k \).
Note without knowing how big \( C \) is there might not be an \( R_k \) for small \( k \) (for instance if \( C>e^* \)
f(0) then there is no \( R_0 ) \). But this isn’t a problem since the roots that could be arbitrarily close
to the origin are still finite in number - we can solve \( M(R_k) = k \) for \( \text{some k} \).

Algebra gives \( R_k = (k\log_2 - b)^\lambda \) where \( b = \log(C/|f(0)|) \)
Then our sum is less than \( \text{(finite number of finite numbers)} + \text{Sum[const / (k log2 - b)^\epsilon]} \)
which is finite for any \( \epsilon > 0 \).
I proved it pretty cleanly I think. I’ll share the pdf with my TeX’d solution (among others).

10. Show \( H \) is a closed subspace of \( L^2(D,\mu) \) since a convergent sequence in \( L^2 \) is bounded
and thus you can show the sequence is a normal family so every subsequence has a
subsequence converging uniformly on compact subsets of the disk to a holomorphic function,
since the subsequence also converges in \( L^2 \) these must be the same function, blah blah
blah done.
11. Use the argument principle by looking at the image of circles with radius bigger than \( r \) under the function \( f \). These must be Jordan curves, so use the Jordan curve theorem to show its winding number around any point in the plane is 1 or 0.
12. Perhaps this has something to do with CA3 on Winter 2004?

By conformal it means analytic right? How about extending it to \( 2Q \) by reflecting horizontally and vertically across the boundary of \( Q \). Then you've got something that extends to be doubly periodic on the plane with period parallelogram \( 2Q \). It's analytic on the closed period parallelogram by Morera.

or just straight up apply Schwarz reflection twice (pre and post compose with some shifts/90 degree rotation so that the imaginary part is zero on the boundary in question). So it's entire and bounded so it's constant so it doesn't map \( Q \) onto \( R \) contradiction.

(When you do Schwarz reflect you reflect the range too, so technically the function you obtain isn't periodic or bounded).

Yeah I realized this: you can either try to reflect in a way that makes it periodic, or schwarz reflect. I don't think the former is possible in a way that makes it analytic. How about this: Schwarz reflect to extend it to an entire function that is univalent, and hence a linear function (for the form \( az+b \)) by a different qual problem, and hence doesn't map \( Q \) 1-1 onto \( R \). [THIS]

Shorten this a bit: don't have to do that old qual problem all over again because the essential singularity case is already cut out - \( f(z) \) goes to infinity as \( z \) does, so \( f \) has a pole at infinity, so it's a polynomial, and if it's of degree higher than 1 it's not 1-1 by the fundamental theorem of algebra.

We're using the fact that boundaries go to boundaries, but this follows from corners going to corners and homeomorphism right?(yeah it should)

Suppose part of an edge \( E \) gets mapped to the interior. Since this is a homeomorphism and its endpoints go to the corners, \( R\setminus f(E) = f(Q \setminus E) \) is disconnected. But \( Q \setminus E \) is connected - contradiction.

Or is this problem supposed to drop out of looking at lines and angles?

Riemann Mapping theorem and cristoffel stuff?(maybe?)

**Fall 2009:**

1. \( \{ (1/2+1/n) \phi_n \} \) \( \Rightarrow \) for an infinite ON set
2. multiply and divide by \( (1+n^2+2+m^2) \) in the \( \ell^1 \) norm of the coefficients of \( v \) then use cauchy schwarz.

\[
m(\{ f<\alpha+1 \}) = \int_{\{ f>\alpha+1 \}} dx \leq \int_{\{ f>\alpha+1 \}} f(x) - \alpha \, dx
\]

3. (a)

\[
\leq \int_{\{ f>\alpha \}} f(x) - \alpha \, dx = \sum_{a_i} \int_{b_j} f(x) - \alpha \, dx
\]
where we used that \( \{ f > \alpha \} \) is open by fcts so it is a union of disjoint open intervals and since the endpoints of these intervals are in the boundary of \{ f > \alpha \} f(a_j) = f(b_j) = \alpha = \min(f(x)) on the interval \( (a_j, b_j) \).

(b) rewrite the integral in terms of the distribution function, do some changes of variables and use the estimate from part (a).

4. We did this one in 245B, the proof is also similar to the one in folland’s section on differentiation theorems.

5. put a fat cantor set in each open set from a countable base for topology, you have to make their measures shrink in such a way that they will not fill up any interval. For example let \( \mathcal{U}_n \) be a countable base for topology for each \( n \) let \( C_n \) be a fat cantor set in \( \mathcal{U}_n \) with positive measure that is less than \( \frac{1}{2^n n!} \). Let \( I_{n,j} = [2^{-n} j, 2^{-n} (j+1)] \) with \( n \geq 0 \) and \( j \) an integer, ergo the diadic intervals of length less than or equal to \( 1 \). Then we will define a subset \( C_{n,j} \) of \( I_{n,j} \) inductively as follows. Let \( C_{0,j} \) be a fat cantor set such that \( 0 < m(C_{0,j}) < 1/2 \), for all \( j \). (We need only define \( C_{n,0} \) for each \( n \) and then let \( C_{n,j} \) be the appropriate translation.) Note that \( m(I_{n,j} \cap C_{0,k}) < m(I_{n,j}) / 2 = 2^{-n-1} \) for all \( j \) and \( k \) by construction of the fat cantor set. Next define \( C_{1,j} \) as a fat cantor set such that \( 0 < m(C_{1,j}) < 1/8 \) and \( m(I_{1,j} \cap (C_{1,k})) \) is 0. This is possible because the quantity on the right is positive: firstly \( I_{2,j} \) intersects exactly one \( I_{1,k} \) (by the nature of the embedding of diadic intervals) and therefore at most one \( C_{1,k} \), by our above note this means \( m(I_{2,j} \cap (C_{1,k})) < 1/8 \). Inductively choose \( C_{n,j} \) to be fat cantor sets with \( 0 < m(C_{n,j}) < 2^{-(2n+1)} \) for all \( n \) and \( k \) by construction of the fat cantor set.

[Not actually sure if this works... computation is incredibly annoying]

The following might be cleaner: Take all rationals with prime denominator and call them “odd” if the numerator is odd and “even” otherwise. Then enumerate them odd-even-odd-even-etc. Now if the enumerated list is \( \{ a_j \} \), create a set \( E \) by adding the ball of radius 1/2 around \( a_1 \), removing the ball of radius 1/4 around \( a_2 \) (if there’s anything to remove), adding ball radius 1/8 around \( a_3 \), removing ball radius 1/16 around \( a_4 \), and so on.

\( E \) has full measure intersection with no interval: given an interval \([b,c]\) there exists an even \( a_j \in [b,c] \) with its removed ball contained in \([b,c]\) and the measure of this removed ball is large enough that it could never all be ‘added back’ in the construction of \( E \).

\( E \) has positive measure intersection with every interval by an analogous argument.

And this is the nicest: http://www.math.ucla.edu/~jheilrun/course_files/Fall%202005/Math%20245A%20-%20Real%20Analysis%2828Rudin%29%20Well-distributed%20measurable%20sets.pdf

\[ h(r e^{i \theta}) \frac{d \theta}{2 \pi} \text{are precompact in the weak star topology of} \ C(\partial D)^* \] since they are bounded in norm using mean value property of harmonic fcns so use Banach-Alaoglu) so take a subsequential limit \( r_n \to 1 \) to get a measure on \( \partial D \) which has the right property, to check the
property it is useful to note that the poisson kernel is in $C(\partial D)$.

7. (a) probably easiest to define it as a bounded linear operator on $H$ with bounded inverse such that $S^{-1} = S^*$. 
(b) this is the same as previous qual problems just use the fact that $\|S^*\| = \|S\| = 1$ and the rewriting $S - \lambda I = S(I - \lambda S^*)$, now $\|\lambda S^*\| < 1$ for $\lambda \in E$.
(c) I think you can just show that it has a power series using the series for $(S - \lambda I)^{-1}$ to prove the real part is positive let $y = (S - \lambda I)^{-1}x$ then 
$\langle (S + \lambda I)(S - \lambda I)^{-1}x,x \rangle = \langle (S + \lambda I)y,(S - \lambda I)y \rangle = \|Sy\|^2 - |\lambda|^2\|y\|^2 + 2i\, \text{Im}(\lambda < y,Sy>)$
then use $|\lambda| < 1$ and $S$ unitary to get the result.

$0 = f(z_1) - f(z_2) = \int_0^1 f'(z_1 + t(z_2 - z_1))(z_2 - z_1)dt$

8. (a) if you expand this into real and imaginary parts and use the assumption it should become 
$a(z_1 - z_2) = ib(z_1 - z_2)$, where $a$ and $b$ are real and $a > 0$, so $\Im(z_1 - z_2) = 0$.
(b) $z^a(3/2)$

9. (a) if it had no pole in the unit disk then you can show by periodicity that it is entire and bounded and thus constant, so it must have at least one pole in the disk. So by assumption it must have exactly one pole in the disk. (note that the plane is contained in the union of the closed balls of radius 1 about the points of the form $\sqrt{2}(n_1 + i n_2)$.

(b) its contour integral around the square with sides of length $\sqrt{2}$ is zero (just choose a square which doesn’t have a pole on the bdry) so the residue at the pole is zero so it must have order > 1. (may have to shift stuff around)

10. First map with $\Exp[-iz]$ then send to the sphere in the normal way
12. compare $f(z)$ with $1/\sin(z)$ you can show that $f(z)\sin(z)$ is bdd (by considering $x=n^*\pi$). along those vertical lines, $f(z)\sin(z)$ is bounded and since f has no essential singularities, also as z approaches $n^*\pi$ along a vertical line $f(z)\sin(z)$ remains bounded and so it cannot have a pole at this point. The singularities of $f(z)\sin(z)$ are then removable, so apply maximum modulus on rectangles with $|y|=1$ and $x=n^*\pi$ to get f bounded there, and bounded for $|y|>1$ by the assumed bound on $|f|$, and therefore extends to a bdd entire function and is thus constant. Then $f(z)$ has the same poles as $1/\sin(z)$ at all points of the form $n\pi$.

Spring 2010:
1. (a) Standard proof (b) Typewriter sequence
2. 246A or 246B
3. Standard proof!
4. (a) Just proof Poisson-Jensen (b) If this set had positive measure then as we increase r the integral should plummet to negative infinity, contradicting our bound from below
5. (a) Approximate by $C_c$ functions and use density argument (b) Convolution trick
6. Same as 7 on Spring 2009
7. Standard
8. Use argument principle and the fact that F (and therefore $F'$) are doubly periodic
9. (a) Let \( x^n \) be a sequence (of sequences) in \( A \), we can produce (by diagonalization) an element \( y \) such that \( x^{n \downarrow} \) converges to \( y_\downarrow \) in \( n \downarrow \) for each \( m \). So \( \sum_{n=1}^N n |y_n|^2 \leq 1 \) holds for all \( N \) by pointwise convergence and therefore it holds as we let the sum tend to infinity, thus \( y \) is in \( A \). To see that \( x^{n \downarrow} \) converges to \( y \) in \( l^2 \), let \( N \) be such that \( n = \sum_{n=N+1}^{\infty} \frac{1}{n^2} < \frac{\varepsilon}{8} \). Then
\[
\sum_{m=1}^{\infty} |x^{n m}_m - y_m|^2 = \sum_{m=1}^{N} |x^{n m}_m - y_m|^2 + \sum_{m=N+1}^{\infty} n^2 |x^{n m}_m - y_m|^4 \frac{1}{2} \left( \sum_{n=N+1}^{\infty} \frac{1}{n^2} \right)^{1/2} \\
\leq \sum_{m=1}^{N} |x^{n m}_m - y_m|^2 + \left( \sum_{m=N+1}^{\infty} n |x^{n m}_m - y_m|^2 \right)^{1/2} \frac{\varepsilon}{8} \\
\leq \sum_{m=1}^{N} |x^{n m}_m - y_m|^2 + (4 \sum_{m=N+1}^{\infty} n |x^{n m}_m|^2 + n |y_m|^2)^{1/2} \frac{\varepsilon}{8} \\
\leq \sum_{m=1}^{N} |x^{n m}_m - y_m|^2 + (8)^{1/2} \frac{\varepsilon}{8} \sum_{m=1}^{N} |x^{n m}_m - y_m|^2 + \varepsilon
\]
So we need only take \( n \) large enough so that \( |x^{n m}_m - y_m| \) is small for each \( 1 \leq m \leq N \).

(b) Use C-S on the integral.

10. First show that \( \{ z \in \Omega : \sup_{u \in \Omega} u(z) < \infty \} \) is open and closed via Harnack’s inequality. Since \( z_0 \) is in this set we know it must be all of \( \Omega \). Take a compact set, cover it will balls that won’t leak out of \( \Omega \), reduce to a finite subset and apply Harnack again.

11. (a) Interpolate between bounds on the \( L^p \) and \( L^{\infty} \) norms.

(b) Let \( \frac{1}{1 - |x|} \leq \phi(x) \leq \frac{1}{1 - 2|x|} \) be continuous. Then consider \( f(x) = 1/(1 + |x|) \). It is in \( L^2 \) but \( f \) convolved with \( \phi \) shouldn’t be in \( L^1 \).

12. For the boundedness rewrite the difference quotient as an integral of the derivative along a curve (with an appropriate \( f \) substituted for \( F \)) and then the fact that \( \{ f \in L^1 : f \text{ is one of the family of analytic functions described} \} \) are bounded uniformly on compact subsets of the disk. Then in order to show that any two subsequential limits are the same you should show that the difference quotients of functions in the family at a particular point converge uniformly (over all functions in the family) to the value of their derivatives at that point.

13. “If” direction is easy. “Only if” and the following. Since \( X' \) is separable, \( X \) is (see some previous qual I can’t recall). First reduce to a subsequence such that \( A(x_n) \) converges to \( y \) in \( Y \). Then by Banach-Alaoglu, a further subsequence \( x_{n \downarrow} \) converges in the weak star topology (that is as elements of \( (X')^* = X \)). Say the limit is \( w \), then \( A(x_{n \downarrow} - w) \) converges to \( y - A(w) \).

Then let \( g \) be in \( Y' \), so that \( h \circ g \) composed with \( A \) is in \( X' \) and so \( h(x_{n \downarrow} - w) \) converges to \( 0 \) for all \( g \). But this means \( A(x_{n \downarrow} - w) \) converges weakly to zero and also strongly to \( y - A(w) \). Hence \( y = A(w) \) (by Hahn-Banach since this element being non-zero means we could construct a linear functional on which it isn’t zero), so if we take \( r_{n \downarrow} = x_{n \downarrow} - w \), then we have the desired result.
Slightly different endgame: suppose $Ax-y$ is not zero. Then there exists $g$ in $Y^*$ st $g(Ax-y)$ is not zero. But then do the above and you get $g(Ax-y)=0$.

**Fall 2010**
1. Lebesgue dominated convergence theorem (use fatou on $2|g|-|f_n-f|$)
2. Easiest way to do this in general is note that convex functions have a supporting hyperplane i.e. for each $t_0$ there is some $v$ such that $\varphi(t) \geq \varphi(t_0) + v \cdot (t-t_0)$, take $t = f(x)$, and $t_0 = \int f(x) dx \quad \text{then integrate on } [0,1]$.
3. For $f_n \in L^\infty([0,1]) \cap L^1([0,1])$ which are uniformly bounded in $L^\infty$ norm converging weak star in Linfty implies weak convergence in L1. If $\int f_n \varphi dx$ converges to $\int g \varphi dx$ for some $g \in L^\infty([0,1])$ and all $\varphi$ in $L^1([0,1])$ then clearly it also converges on $L^\infty([0,1]) \subset L^1([0,1])$ and $g \in L^1([0,1])$ as well, so the fn converge to $g$ in weak L1. Unfortunately it is kind of annoying to prove this convergence. First of all note that:
   $$\frac{1}{n} \int f_n(x) dx = \sum_{j=0}^{n-1} \frac{(j+1)/n}{j/n} \int f_n(x) dx = \sum_{j=0}^{n-1} \frac{\exp(\sin(2\pi y))}{j/n} \frac{dy}{n} = \frac{1}{n} \int f(x) dx = \alpha$$
   by the periodicity of sin,
   $$\int \frac{(j+1)/n}{j/n} \exp(\sin(2\pi nx)) dx = \alpha / n$$
   moreover
   $$\int \frac{(j+1)/n}{j/n} \exp(\sin(2\pi nx)) dx = \alpha / n + O(1/m)$$

   work for fixed $n$ going to infty thus for all indicator functions of intervals of the form $[j/n,(j+1)/n]$
   $$\int f_n(x) \chi_{j/n < x < (j+1)/n} dx \to \int \alpha \chi_{j/n < x < (j+1)/n} dx$$
   and the finite linear combinations of these indicator functions are dense in $L^1([0,1])$ so we can deduce the weak* convergence. You can also do this for continuous functions, which are also dense. Riemann sums pop out nicely.

4. We will use the characterization of the $L^p$ norm $\|f\|_p = \int_0^\infty p^{p-2} f(t)^p dt$ (derived using the ID $|f|^p = \int p^{p-2} f_{t \leq |f|} dt$). The idea is a variation on splitting things up into small and big parts $f_1(x) = f(x) \chi_{x > t/2}$ then $T(f)(x) \leq T(f_1)(x) + T(f \chi_{x < t/2})(x) \leq T(f_1)(x) + \|T(f \chi_{x < t/2})\|_\infty \leq T(f_1)(x) + \|f \chi_{x < t/2}\|_\infty \leq T(f_1)(x)$. So
   $$\|Tf\|_p = \int_0^\infty p^{p-1} |\{Tf > t\}| dx \leq 2 \int_0^\infty p^{p-2} \int |f| \chi_{\{f > t/2\}} dx dt$$
\[ 2p \int \left| f \right| \int_0^{\frac{2}{p-2}} t^{p-2} dt \ dx = 2p \frac{p}{2p-1} \left| f \right| L^p \]

Other way is to use Marcinkiewicz interpolation theorem.
5. (a) use the hint and density of trig polynomials (similar to 4 on Winter 2007)
(b) Let \( F \) be a closed subset of the torus \( g(x+Z) = \chi_F(x+Z)-m(F) \) can be written as the inifimum of a sequence of continuous functions which converge in \( L^1 \) norm. Just bound
\[
\frac{1}{N} \sum_{n=0}^{N-1} g(n\alpha+Z) \leq \frac{1}{N} \sum_{j=0}^{N-1} f_j(n\alpha+Z) ,
\]
and take lim sups on both sides in \( n \) and \( j \).
(This shows the lim sup is <= zero, need to then get function \( h_1 \) which converge upwards to \( g \) to get \( \liminf \geq 0 \).
Can't we take limits rather than \( \limsup / \liminf \) by Dominated convergence theorem?
6. (a) multiply and divide by \((1+|k|^2)^{1/2}\) and then use cauchy schwarz
(b) \( \widehat{g}(k) = \frac{1}{(1+|k|^2)} \)
c) use the same argument as in part (a) but now with the fact that \( \widehat{f}(0) = \mathbb{C} \) and the norm is bounded by one to get a better bound on \( \text{Re} L(f) \) then show this bound is achieved by \( (g-1)\|g-1\| \) (g from part (b)) to show that the maximum is achieved. To show that this
unique note that if \( ||f||<1 \) then \( \text{Re} L(f) \geq \text{Re} L(f) \) so any maximizer must have \( ||f||=1 \).
If \( \varphi \) and \( \psi \) maximize and are not equal and \( f = (\varphi + \psi)/2 \) then
\( \text{Re} L(f) = (\text{Re} L(\varphi) + \text{Re} L(\psi))/2 \) maximizes as well but
\( ||f||^2 = \sum (1+|k|^2) \| \varphi(k) + \psi(k) \|^2 / 4 < 1 \) (since cauchy schwarz is strict when the vectors are not linearly dependent) which contradicts maximizers being on the boundary, so phi and psi must have been the same.
7. Use Morera's theorem for rectangles.
8. L(a) = a where a is a constant function. since L(1) = L(1)^2 and L not identically zero implies L(1) = 1, thus for all complex numbers a L(a) = L(a^*) = a* L(1) = a. Now suppose L(z) = a not in the disc then L(z-a) = 0 but z-a is never zero in the disc so for all g analytic on the disc L(g) = L(z-a)L(g(z-a))=0 but L is not identically zero by assumption. Similar arguments imply L(f) = f(z) for some z in the disc but not necessarily the same z for each f. Clearly its the same z_0 for all polynomials, also for all functions that have a zero of any order at z_0, L(f)=L((z-z_0)^m h) = 0.
So now for any function 0=L(f(z)-f(z_0)) = L(f(z)) / f(z_0) so L(f(z)) = f(z_0).
Just rewriting this for my own benefit:
L(1)=1. L(a)=a for all a in C.
Let id denote z|->z.
Let z0 = L(id).
z0 must be in D, for if not, id-z0 is holomorphic on D, and for any other g holo on D, L(g)=L((id-z0)g/(id-z0)) = 0 =><=
Now for any f holo on D,
L(f-f(z0))=L(f-f(z0))=L((id-z0)^m h) some h holo on D
= L((id-z0)^m L(h) = 0. So L(f) = f(z0) for all f holo on D.

So it turns out you can prove that L is actually continuous a priori so then you can extend the result for polynomials by density. First note L(g) = L(1^*g)=L(1)L(g) implies L(1)=1 for L not
identically 0. Also for nowhere zero g, 1=L(1)\leq L(g'/g) (sup norm) then \( 1-g/L(g)\geq 0\) i.e. \( 1-g/L(g)\) is invertible so \( 0 \neq L(1-1/L(g)) = 1/L(g)/L(g) = 0\) which is a contradiction. Therefore for all g holomorphic on the disk \( \|g\|_\infty \) replacing g by -g we get the other direction as well.

the topology above (uniform norm) means we aren't considering the full space (which isn't a normed linear space). topology for A is generated by uniform topologies on compact subsets. This is not a problem, for f holomorphic on the disk and B(0,r) compactly contained in the disk and containing the point where polynomials are evaluated, for every epsilon>0 there exists a polynomial P such that \(|f-P|\) (on K)<epsilon so now if we restrict the norm to B(0,r) we get \(|f(z)-f(z)|=|L(f)-L(P)|+|P(z)-f(z)|\) where we are considering the restricted Banach space of bounded holomorphic functions on B(0,r) and with the natural restriction of L.

9. \( g(z) = (f(z)-f(0))/f'(0) \) is injective iff \( f \) is so lets consider WLOG. Now the given

\[
|g'(z)-1| = |\sum_{n=2}^{\infty} n b_n z^{n-1}| < \sum_{n=2}^{\infty} n |b_n| = \sum_{n=2}^{\infty} n |a_n/a_1| \leq 1
\]

condition tells us that for z in the interior of the disk and b_n are the power series coefficients of g. Note that this is telling us that the derivative of g lies in the open ball of radius 1 about the point z=1, i.e. the real part of the derivative is strictly positive. Now we can use the previous qual problem which says a holomorphic function on a convex set with derivative with strictly positive real part is injective. The proof of this involves using the fundamental theorem of calculus along the line segment between two points which violate the injectivity and splitting into real and imaginary parts.

Rouche (on arbitrary balls of radius r<1) implies f' is nonzero on the disk. Then as above (Need more that f' being non-zero to use the above proof, need either real or imaginary part to be either all positive or all negative)

Better: Rouche actually shows f-w is nonzero on the disk for all w in the plane.
(I don't think this works either, bc you can only bound above with |a_1|, not |a_1z-w|.)

10. Suppose there existed a conformal map of A(0,1) to A(1,2) \( \varphi \). Use Riemann removable singularity theorem to extend it to a map of the disk to the closed annulus \( \psi \), then open mapping thm implies the image of the disk is actually contained in the open annulus so since the original mapping was 1-1 and onto now the extended mapping violates 1-1ness only at one pair of points (0 and \( \varphi^{-1}(\psi(0)) \)). This is a contradiction: by the open mapping theorem, disjoint neighborhoods of 0 and \( \varphi^{-1}(\psi(0)) \) are mapped to neighborhoods of \( \psi(0) \) which have nonempty intersection since they are open neighborhoods of the same point contradicting \( \varphi \) being 1-1.

11. just calculate the laplacian of f and \( f^2 \) and you will get that \( \nabla u \cdot \nabla v = c \) where \( u \) and \( v \) are the real and imaginary parts of f respectively. This implies that either f or f bar is holomorphic since it means \( \nabla u = \pm U \nabla v \) where \( U \) is a counterclockwise rotation by \( \pi/2 \).

12. cauchy integral formula/mean value property, a version of this one came up on a lot of previous quals.

Spring 2011:

1. (a) [http://en.wikipedia.org/wiki/Weak_convergence_%28Hilbert_space%29](http://en.wikipedia.org/wiki/Weak_convergence_%28Hilbert_space%29) (b) First note that the \( F_n \) converge pointwise to F:
\[
\left| \int_0^x f_n(t) \, dt - \int_0^x f(t) \, dt \right| \leq \left| \int_0^x \chi_{[0,x]}(f_n(t) - f(t)) \, dt \right| = \langle f_n - f, \chi_{[0,x]} \rangle \to 0
\]

Since \( \chi_{[0,x]} \) is of course an L^2 function. Next, for all \( g \) in L^2 we have \( \langle f_n, g \rangle \) is bounded in \( n \) since they converge to \( \langle f, g \rangle \). So as operators on L^2, they are pointwise bounded in \( n \) and so by the Uniform Boundedness Theorem their operator norms are bounded in \( n \). Since their operator norms are precisely their L^2 norms, we have \( ||f||_2, ||f_n||_2 < M \) for all \( n \). This then implies that the functions are uniformly bounded and equicontinuous:

\[
|F_n(y) - F_n(x)| = \left| \int_x^y f_n(t) \, dt \right| \leq \int_x^y |f_n(t)| \, dt \leq \| f_n \|_2 \leq M \sqrt{y-x}
\]

(for boundedness just let \( x = 0 \)). Hence by Arzela-Ascoli every subsequence has a subsequence which converges uniformly, but as the point-wise limit is \( F(x) \) the uniform limit must be this as well. By a standard argument this implies that the sequence itself converges uniformly to \( F \).

2. Do the most obvious estimate possible (bound sin by 1) and note that \( f \) in L3 implies \( f \) in L1 and the Lebesgue differentiation theorem holds for functions in L1:

\[
g(t) = e^{itx} \, d\mu(x) Re((g(0) - g(t))/t^2) = \frac{1}{t^2} \int 1 - \cos tx \, d\mu
\]

3. Define

\[
= \frac{2}{t^2} \int \sin^2(\frac{tx}{2}) \, d\mu \geq \int -\frac{\pi}{t} / \frac{\pi}{t} / 2 d\mu
\]

where we have used the inequality \( |\sin(y)| = |y| / 2 \) on \([-\pi / 2, \pi / 2] \), then take limits on both sides (using MCT on the right) and it is easy to conclude the result.

4. First note that by Fatou

\[
\int f(x) \log(2 + f(x)) \, dx \leq \sup \int f_n(x) \log(2 + f_n(x)) \, dx < \infty
\]

so that

\[
\int f(x) \, dx = \int f(x) \, dx + \int f(x) \, dx \leq \int f(x) \log(2 + f(x)) \, dx + 2 - e < \infty
\]

. i.e. \( f \in L^1([0,1]) \). Now also note that \( \{f_n \} \cup \{f\} \) is a uniformly integrable family of functions since for any \( \epsilon > 0, a > 0 \) s.t. \( \log(2 + M) > a / \epsilon \). So that

\[
\int f_n \log(2 + f_n) \, dx \geq a / \epsilon \int f_n \, dx
\]

\[\{f_n \geq M\} \]

\[a = \sup \int f_n(x) \log(2 + f_n(x)) \, dx \sup \int f_n \, dx \leq \epsilon \]

and then if

\[\{f_n > M\} \]

\[e > 0 \exists \delta > 0 \text{ s.t. } m(E) < \delta \Rightarrow \sup \int f_n \, dx < \epsilon \]

This allows us to say that given

\[\{f_n \geq M\} \]

\[\sup \int f_n \, dx \leq \epsilon / 2 \]

\[\delta < \epsilon / M \]

\[m(E) < \delta \]

take \( M \) large enough that

\[\{f_n \geq M\} \]

and then for we
\[
\int f_n \, dx = \int_{E \cap \{ f_n > M \}} f_n \, dx + \int_{E \cap \{ f_n \leq M \}} f_n \, dx \leq \epsilon / 2 + M \delta \leq \epsilon
\]

have for arbitrary \(n \in E\). Now ptwise a.e. convergence implies almost uniform convergence on finite measure spaces so given epsilon let A be a subset of \([0,1]\) s.t. fn converge to f uniformly on A and \(M(0,1)\Delta A < \delta\) where delta is chosen as above. Then
\[
\int |f_n - f| \, dx \leq \int_{[0,1] \setminus A} \|f_n + f\| \, dx + \|f_n - f\| A \|_{\infty} \leq 2 \epsilon + \|f_n - f\| A \|_{\infty}
\]

where we can make the infinity norm arbitrarily small by choosing \(n \) large. (A proof of the statement “ptwise a.e. convergence implies almost uniform convergence on finite measure spaces”: since \([0,1]\) is finite, convergence a.e. implies convergence in measure. Fix \(\epsilon > 0\)

, if \(E_n = \{ x \in [0,1] : |f(x) - f_n(x)| > \epsilon \} \), then \(m(\cap E_n) = 0\) by a.e. convergence. Since each \(E_n \subset [0,1]\) we have by continuity from above that \(m(E_n) \to 0\). Take \(N\) large enough
\[
m(E_n) < \delta = \delta \implies \int |g| < \epsilon
\]

so that for all \(n > N\), \(g \in \{f_n\} \cup \{f\}\) implies \(\in E\) for
\[
\int |f - f_n| \, dm \leq \int_{E_n} |f - f_n| \, dm + \int_{E_n^c} |f - f_n| \, dm \leq 2 \epsilon + \epsilon
\]

. We have for all \(n > N\)

5. (a) For every element in \(P(Z)\) we obtain a distinct indicator function, \(|P(Z)|\) is continuum (b) Use part a and the fact that each of these indicator functions are at least a distance 1 away from each other (c) Since \(L^\infty\) is not separable, neither is its dual, but \(L^1\) is separable and thus we cannot have \((L^\infty)'' = (\text{all finite})' = L^1\)

6. Standard

7. Standard

8. (a) \(\{f < \alpha\}\) should be open for all \(\alpha \in [-\infty, +\infty]\).

(b) Take the easier to work with definition here, an upper semicontinuous function \(v\) on a domain \(\Omega\) in the complex plane is said to be subharmonic if for each \(z \in \Omega\) and each ball \(B(z, r) \subset \Omega\)
\[
v(z) \leq \frac{1}{2 \pi} \int_0^{2 \pi} v(z + re^{i\theta}) \, d\theta
\]

we have

\[
v(z) \leq \frac{1}{2 \pi} \int_0^{2 \pi} v(z + re^{i\theta}) \, d\theta \leq \frac{1}{2 \pi} \int_0^{2 \pi} \sup_{\alpha \in A} v^\alpha (z + re^{i\theta}) \, d\theta
\]

we have for each

\(\alpha \in A\), so it is also true for the supremum. (why is it USC? not sure at the moment). As for a counterexample for the infimum of a family of subharmonic functions, On the unit disk consider \(v(z) = \inf_{\alpha \in D} (|z - \alpha| - |\alpha|) = -|z|\) which is not subharmonic (it violates the maximum principle first of all).

(d) note that \(\log \langle A(z) \xi, \eta \rangle\) is subharmonic for each \(\xi\) and \(\eta\) in the unit sphere of
$C^2$ (since $\langle A(z)\xi,\eta \rangle$ is holomorphic plus the Jensen formula) and this family is bounded ptwise by the log of the operator norm of $A(z)$, and $\|A(z)\| = \sup_{\xi,\eta} \log \|A(z)\|_e$ is subharmonic by part b.

$Q_\epsilon = [-\epsilon,1+\epsilon] \times [-\epsilon,\epsilon]$, $B = \{|z|<2\}$, $F(z) = \frac{1}{2\pi} \int_{\partial B} \frac{f(\zeta)}{\zeta-z} d\zeta$.

9. Define $G_\epsilon(z) = \frac{1}{2\pi} \int_{\partial Q_\epsilon} \frac{f(\zeta)}{\zeta-z} d\zeta$ and $Q_\epsilon C$

$G_\epsilon(z) = \frac{1}{2\pi} \int_{\partial Q_\epsilon} \frac{f(\zeta)}{\zeta-z} d\zeta$

where $F$ is analytic on $B$ and $G_\epsilon$ is analytic on $\partial Q_\epsilon$. Then for $z \in B \setminus Q_\epsilon$ we have $f(z) = F(z) - G_\epsilon(z)$. We will show that $G_\epsilon(z) \to 0$ as $\epsilon \to 0$ for each $z \in B \setminus [0,1]$, which will imply $f(z) = F(z)$ on $z \in B \setminus [0,1]$ and hence $f$ extends to be analytic on $B$ (as $F$). For $z \in B \setminus [0,1]$, there is some $\epsilon_0 > 0$ s.t. $z \in B \setminus Q_{\epsilon_0}$. Since the $Q$ epsilon are nested this holds for all smaller epsilon as does the formula for $f$ in terms of $F$ and $G_\epsilon$.

$\text{dist}(z,\partial Q_{\epsilon_0}) = \delta > 0$ $\zeta \in \partial Q_\epsilon$ $\epsilon < \epsilon_0$

so on for all we have $|f(\zeta)| \leq \|f\|_\infty / \delta$ so the vertical components of the line integral in the definition of $G$ epsilon go to zero as epsilon goes to zero.

Let $C_N$ be the $N$th approximation to the cantor set $m(C_N) = (2/3)^N$ (obtained by removing the $2^N$ middle third intervals of $C_{N-1}$) and note that by homotopy (probably should prove this by induction, you can convince yourself it is true by drawing a picture).

$\int_{\partial(C_N \times [-\epsilon,\epsilon])} \frac{f(\zeta)}{\zeta-z} d\zeta$

$\leq \frac{1}{\delta} \|f\|_\infty (2(\frac{2}{3})^N + 2\epsilon \sum_{j=1}^{2^j}) = \frac{1}{\delta} \|f\|_\infty (2(\frac{2}{3})^N + 2\epsilon(2^N+1-1))$

$\eta > 0$ $N$ $(\frac{2}{3})^N < \frac{1}{4\|f\|_\infty \eta \delta}$

Now for any just choose large enough that $\epsilon < \eta \delta$ and then

$\epsilon < \eta \delta < \frac{1}{4\|f\|_\infty (2^N+1-1)}$.

10. The proof of the Riemann mapping theorem or just general intuition for maps of the disk to itself should inspire you to believe that the supremum is achieved by the conformal mapping of omega to the disk which sends $i/2$ to zero and whose derivative at $i/2$ is real and positive. To prove that this is the case lets build up. Let $f: \Omega \to D$ with $f(i/2) = \alpha$ and $g(z) = \varphi_\alpha(f(z))$

$\varphi_\alpha(w) = e^{-i \arg f(i/2)} \frac{w-\alpha}{1-\bar{\alpha} w}$ then also maps omega to the disk and
\[ g(i/2) = 0 \quad \text{and} \quad g'(i/2) = \varphi'(\alpha)f'(i/2) = |f'(i/2)| \frac{1}{1-|\alpha|^2} \geq \Re (f'(i/2)) \]

and is real and positive. So it suffices to consider \( f \) which map \( i/2 \) to 0 and have real and positive derivative at \( i/2 \). Now let \( \psi: \Omega \to D \) be the conformal mapping of \( \Omega \) to the disk with the properties above. This exists by the Riemann mapping theorem (and is unique), but we will be constructing it explicitly anyway, first lets show that this achieves the supremum in the problem though. Let \( f: \Omega \to D \) be holomorphic \( i/2 \) to 0 and have real and positive derivative at \( i/2 \).

Consider \( g(z) = f(\psi^{-1}(z)) \) which maps the disk to itself and sends zero to zero. By Schwarz lemma \( |g'(0)| \leq 1 \) so
\[
\Re f'(i/2) = |f'(i/2)| \leq 1 / |\psi^{-1}'(0)| = |\psi'(i/2)| = \Re \psi'(i/2)
\]
hence \( \psi \) must achieve the supremum. In order to finish the problem you need to construct \( \psi \) and calculate its derivative at \( i/2 \), this part is standard if tedious. (I'm not sure what properties we gave \( \Psi \) in the first place? If it was that it sends \( i/2 \) to 0 and has a real positive derivative at \( i/2 \) then why did you consider \( f \) with the same properties I had already proven that any maximizing \( f \) would have to send \( i/2 \) to zero and have positive real derivative at \( i/2 \), however I had not proven that a maximizing \( f \) need be conformal. \( \Psi \) is chosen to send \( i/2 \) to zero have positive real derivative at \( i/2 \) AND also be conformal. You cannot reverse the inequality unless \( f \) is conformal also because \( f \) does not necessarily have a differentiable inverse which satisfies the inverse function theorem at 0, however it is true that any two conformal mappings that send \( i/2 \) to zero have positive real derivative at \( i/2 \) are the same map. This was the function I obtained:
\[
\psi(z) = i \frac{2z-i}{2z+i}, \text{ it has } \psi'(i/2) = 1
\]

11. part of the proof of the prime number theorem, it's in Gamelin's complex analysis book.

12. Consider a quasi-key-hole contour (you should have all the components be either segments of rays from 0 or arcs of circles around 0) \( g \) which avoids (-infinity,0] and let \( f(z) = \log(z) - t/z \). Then the number of zeros is equal to the winding number of \( g(f(z)) \) about zero. By carefully plotting this contour one can determine that there is exactly one solution, no matter the value of \( t \).

(This section is for useful/interesting Folland problems)

**Folland:** (Chapter.Exercise_Number)

5.17 If \( f \) is bounded then it is continuous. Since \( \{0\} \) is a closed set in a normed vector space we know \( M := f^{-1}(\{0\}) \) is closed by continuity. Conversely, suppose this set is closed. If \( f=0 \) then it is bounded and we are done. Otherwise let \( x \in X \setminus \text{backslash M} \) be such that \( f(x) \) is not equal to zero. Then it is easy to see that \( X = M + Cx \) (where 'C' is the complex numbers).

Suppose towards a contradiction that \( f \) is not bounded and let \( z_n \) be such that \( |z_n| \) and \( |f(z_n)| > n \) for each \( n \). We can write \( z_n = y_n + a_n x \) for \( y_n \) in \( M \) and \( a_n \) in \( C \). Realize that \( z_n / |f(z_n)| \) converges to
zero, and that \(|f(z_n)|=|a_n|\) so \(\{a_n/|f(z_n)|\}\) is a subset of the unit sphere in \(\mathbb{C}\). Consequently we can find a convergent subsequence (which we shall still denote with the subscript \(n\)) with limit \(a\). But this implies \(y_n/|f(z_n)|\) converges as the difference of two convergent sequences. Since \(M\) is closed we know this limit, \(y\), is in \(M\). So taking the limit of \(z_n/|f(z_n)|=y_n/|f(z_n)|+xa_n/|f(z_n)|\) we obtain \(0=y+ax\), or \(y=-ax\). Since \(a\) is non-zero (each \(a_n/|f(z_n)|\) had norm one) this implies \(x\) is in \(M\), a contradiction.

5.18(a) We can produce a bounded linear functional \(f\) which is identically zero on \(M\) and \(f(x)=\|x+M\| (>0\ since \(M\) is closed). Suppose \(y_n+a_nx \in M + CX\) converges to some point \(z\) in \(X\).
\[
\lim_{n \to \infty} f(y_n + a_n x) = \lim_{n \to \infty} a_n f(x)
\]
which implies that the \(a_n\) converge to some \(a\). But then the \(y_n\) also converge and since \(M\) is closed this limit, \(y\), is in \(M\). Thus \(z=y+ax\).

5.33 We have a sequence of positive numbers \(\alpha_n\) such that \(\sum \alpha_n |c_n| < \infty\) \iff \(\{c_n\}\) is bounded. Define a linear transformation \(T\) from bounded sequences to \(l^1\) by \(Tf(n) = \alpha_n f(n)\). Then \(T\) is bounded with \(\|T\| = \sum \alpha_n\). \(T\) is also clearly injective since the only preimage of zero is necessarily zero (all the \(\alpha_n\) are positive). \(T\) is also onto: given \(g\) in \(l^1\) let \(f(n) = g(n) / \alpha_n\). We need \(f(n)\) to be bounded, but since \(\sum \alpha_n |f(n)| = \sum |g(n)| < \infty\) we know this is the case by hypothesis. So \(T\) is an open map. Note that finite sequences are dense in \(l^1\) but not in the space of bounded sequences (with the uniform norm) since \(1:1(n)=1\) for all \(n\) for example has a distance of 1 away from any finite sequence. This contradicts \(T\) being an open map since \(T(B(1,1/2))\) should contain an open set around \(T(1)\). This is not the case because we have finite sequences which converge to \(T(1)\) and whose preimages aren’t finite sequences (ergo are in \(T(B(1,1/2))\)).

5.36(b)

5.37 We use the closed graph theorem: suppose \(x_n \to x\) in \(X\) and \(T(x_n) \to y\) in \(Y\). Then since \(f \in Y^*\) and \(f \circ T \in X^*\) are continuous, we know the limit of \(f \circ T(x_n)\) is both \(f \circ T(x)\) and \(f(y)\). Since the linear functionals on \(Y\) distinguish points we must have \(T(x)=y\).

7.21 Define a partial ordering on \(M(X)\) by \(\mu \leq \nu\) if and only if \(E_\mu \subseteq E_\nu\), where
\[
E_\mu = \{ \alpha \in A : \int f_\alpha d\mu = c_\alpha \}
\]
Let \(\mu_1 \leq \mu_2 \leq \cdots\) be an increasing chain in \(M(X)\) by 5Banach-Alaoglu the \(\mu_n\) have a subsequential weak star limit \(\mu_\infty\) and if \(\alpha \in E_\mu\) then for all \(j \geq n\) we have that \(c_\alpha = \int f_\alpha d\mu_j\) so it is also true for the weak-star limit \(\mu_\infty\) i.e. \(E_\mu_n \subseteq E_\mu_\infty\) or \(\mu_n \leq \mu_\infty\), i.e. the chain has an upper bound. Then Zorn’s lemma implies \(M(X)\) has a maximal element lets call it \(\nu\). Suppose there exists some \(\alpha_0 \in A\),
\( \alpha_0 \notin E_\nu \). If \( f_{\alpha_0} \) is not in the closed linear span of \( \{ f_\alpha : \alpha \in E_\nu \} \) (call this subspace \( N \)) then by Hahn Banach there is a linear functional (and hence a radon measure \( \sigma \)) which is zero on \( N \) and \( \int f_{\alpha_0} d\sigma = c_{\alpha_0} \) then \( E_\nu \subset E_{\sigma + \nu} = E_\nu \cup \{ \alpha_0 \} \) where the containment is strict hence \( \nu < \nu + \sigma \) contradicting the maximality of \( \nu \). If \( f_{\alpha_0} \) is in \( N \) then

\[
    f_{\alpha_0} = \sum_{i=1}^{\infty} b_i f_{\alpha_i} \quad \text{where} \quad f_{\alpha_i} \in N \quad i = 1, 2, \ldots
\]

now let \( B_i = \{ \alpha_{0}, \ldots, \alpha_i \} \) as before use Banach Alaoglu on the \( \mu_{B_i} \) to get \( \mu \) with \( E_\mu = \{ \alpha_0, \alpha_1, \ldots \} \)

so then

\[
    c_{\alpha_0} = \int f_{\alpha_0} d\mu = \sum_{i=1}^{\infty} b_i c_{\alpha_i} = \int f_{\alpha_0} d\nu \quad \text{so} \quad \alpha_0 \in E_\nu
\]

which is a contradiction.

**Testable Standard Theorems:**

**Real:**

- Monotone Convergence theorem
- Fatou's Lemma
- Dominated Convergence theorem
- Vitali convergence theorem
- Lebesgue Diff. theorem
- Hardy-Littlewood Maximal function weak type \((1,1)\) bound
- Lusin’s theorem
- Parts of Monotone diff. theorem
- Egoroff
- Conv. in measure implies a.e. converg of subsequence
- Hahn Decomposition

- Radon-Nikodym
- Closed graph theorem
- Open Mapping theorem
- Uniform boundedness
- Holder
- Minkowski's integral inequality
Jensen’s inequality
Young’s inequality
Vitalli-type covering Lemma
Strict containments of Borel sets, Lebesgue sets, arbitrary sets
Riemann-Lebesgue Lemma, sweet proof, see Will
Banach Alaoglu
Weak convergence implies norm of limit is less than liminf of norms
Unique element of minimal norm in closed convex set in hilbert space
Unique projection onto closed subspace
Every linear functional on a hilbert space is an inner product
$L^p$ spaces are complete
Special cases of $L^q$ being dual of $L^p$
Fourier inversion formula when $f$ and $\hat{f}$ are in $L^1$.

**Complex:**

Goursat
Cauchy Integral formula and Cauchy integral theorem equivalence
Morera
Schwarz reflection
Schwarz lemma
Harnack’s princple, inequality
Paineve
Open mapping theorem
Maximum modulus principle
Loowevill (Lou Evil)lalwls
Casorati-Weierstrass
Parts of Riemann mapping theorem
Runge’s theorem
Rouche’s theorem
Argument principle
Finitely connected domains
Hurwitz (Herr Witz)
Equivalence of definitions of subharmonic functions on balls versus arbitrary open subsets
Perron process
Mittag-Leffler
Weierstrass product theorem
Poisson-Jensen formula
Monodromy theorem
Picard’s theorems
Elliptic modular functions