Problem: Let $\Omega$ be an open and bounded subset of $\mathbb{R}^2$, with sufficiently smooth boundary.
Consider the problem
\[
-\frac{\partial}{\partial x}((1 + x^2 + y^2)u_x) - u_{yy} = f \text{ in } \Omega,
\]
\[
(1 + x^2 + y^2)u_x n_x + u_y n_y + \lambda u = g \text{ on } \Gamma = \partial \Omega,
\]
where $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$, $\vec{n} = (n_x, n_y)$ is the outward unit normal to $\partial \Omega$, and $\lambda \geq 0$ is a constant.
(a) For what value(s) of $\lambda$, additional conditions on $u$, $f$ and $g$ are necessary to insure that a weak variational formulation of the problem has one, and only one solution? Give those additional conditions and explain why these are needed.
(b) For $\lambda \geq 0$, give weak variational formulations of the problem (with additional conditions whenever necessary), and show that these formulations have unique solutions.
(c) In the case $\lambda > 0$, describe a FE approximation using $P_1$ elements, and a set of basis functions such that the corresponding linear system is sparse and of band structure. In particular show that the corresponding finite dimensional problem has a unique solution.