Problems #2

[1] Recall that the one-dimensional Discrete Fourier Transform is

\[ F(u) = \sum_{x=0}^{M-1} f(x) e^{-2\pi iux/M}, \]  

for all \( u = 0, 1, 2, \ldots, M - 1 \).

(a) Assuming the formula above, show the identity

\[ f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{2\pi iux/M}, \]  

using the following orthogonality of exponentials

\[ \sum_{u=0}^{M-1} e^{-2\pi iuy/M} e^{2\pi iux/M} = \begin{cases} M & \text{if } x = y \\ 0 & \text{otherwise}. \end{cases} \]

(b) Show now the converse of (a): assume given \( f(x) \) by (2) function of \( F(u) \), and prove the identity (1) for \( F(u) \) (use the same orthogonality of exponentials).

[2] Recall that the 2D Fourier transform in the continuous case is

\[ F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-2\pi i(ux + vy)} dxdy. \]

Show that the transformation \( \mathcal{F} : f \mapsto F \) is a linear transformation.

[3] Compute in continuous variables the Fourier transform of the function

\[ f(x) = \begin{cases} A, & \text{if } 0 \leq x \leq K, \\ 0, & \text{otherwise}, \end{cases} \]

where \( A \) and \( K \) are positive constants. Evaluate \( F(0) \).

(recall that \( F(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi iux} dx \)).

[4] Consider again the 2D continuous Fourier transform defined above, and its inverse defined by

\[ f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{2\pi i(ux + vy)} dudv. \]

Show that if the transform \( F(u, v) \) is real and symmetric, i.e. if

\[ F(u, v) = F(u, v) = F(-u, -v) = F(-u, -v), \]

then the corresponding spatial domain function \( f(x, y) \) is also real and symmetric.
We first recall here the discrete Fourier transforms and their inverses in one and two dimensions.

1D DFT: \[ F(u) = \sum_{x=0}^{M-1} f(x)e^{-2\pi i \frac{ux}{M}}, \quad \text{for } u = 0, 1, 2, ..., M - 1. \]

1D IDFT: \[ f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u)e^{2\pi i \frac{ux}{M}}, \quad \text{for } x = 0, 1, 2, ..., M - 1. \]

2D DFT: \[ F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)e^{-2\pi i \left( \frac{ux}{M} + \frac{vy}{N} \right)}, \quad \text{for } u = 0, 1, 2, ..., M - 1, \ v = 0, 1, 2, ..., N - 1. \]

2D IDFT: \[ f(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v)e^{2\pi i \left( \frac{ux}{M} + \frac{vy}{N} \right)}, \quad \text{for } x = 0, 1, 2, ..., M - 1, \ y = 0, 1, 2, ..., N - 1. \]

[5] (a) Show in discrete variables that
\[ \mathcal{F} \left( f(x, y)e^{2\pi i (u_0 \frac{x}{M} + v_0 \frac{y}{N})} \right) = F(u - u_0, v - v_0), \]
where \( F = \mathcal{F}(f) \).

(b) Using (a), deduce the formula used in shifting the center of the transform by multiplication with \((-1)^x+y\), when \(u_0 = M/2\) and \(v_0 = N/2\), with \(M\) and \(N\) even positive integers.

[6] Show the translation property
\[ \mathcal{F} \left( f(x - x_0, y - y_0) \right) = F(u, v)e^{-2\pi i (x_0 u/M + y_0 v/N)}, \]
where \( F(u, v) = \mathcal{F}(f(x, y)) \).

[7] Consider the linear difference operator \( g(x, y) = f(x + 1, y) - f(x, y) \). Obtain the filter transfer function, \( H(u, v) \), for performing the equivalent process in the frequency domain.

[8] Prove the validity of the discrete convolution theorem in one variable (you may need to use the translation properties).

[9] Assume that \( f(x) \) is given by the discrete IFT formula in one dimension. Show the periodicity property \( f(x) = f(x + kM) \), where \( k \) is an integer.

[10] Consider the two-dimensional continuous case.
(a) Recall the Inverse Fourier transform formula in 2D (express \( f(x, y) \) function of \( F(u, v) \)).
(b) Assume \( f \) is twice differentiable. Using (a), find the Fourier transform of the mixed partial derivative \( \frac{\partial^2 f}{\partial x \partial y} (x, y) \), function of \( F \). 

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