EXAMPLE 2.7.4  Investment period not equal to one year

Problem: Mohammed had $20,000 in his investment account on August 15, 1999. On August 15, 2000 his balance was $21,200 and he deposited an additional $5,000, giving him a new balance of $26,200. On August 15, 2001, Mohammed’s account had a balance of $27,300. Assuming that there are no other contributions to the account, find the annual time-weighted yield and note that it is very close to the annual dollar-weighted yield.

Solution. The balance grows as follows:

\[
\begin{align*}
&\text{Balance} \\
\$20,000 &\rightarrow \$21,200 \rightarrow \$27,300 \\
&\text{Deposit} \rightarrow \$26,200
\end{align*}
\]

Therefore \( i_{tw} = \left[ \left( \frac{27,300}{21,200} \right) \left( \frac{26,200}{27,300} \right) \right]^{\frac{1}{2}} - 1 = 0.05053765. \)

On the other hand, the dollar-weighted yield \( i \) satisfies the equation of value:

\[
\$20,000 (1 + i)^{10} + \$5,000 (1 + i)^{11} = \$27,300.
\]

This equation may be solved using the quadratic equation, and \( i = 0.05 \) is the only positive yield rate.

EXAMPLE 2.7.5  Time-weighted yield less than dollar-weighted yield

Problem: Assume Mr. Haywood notices that although the “Tomorrow Fund” has an excellent performance history, it performed less well when the price of gasoline experienced a sharp rise. He decides to invest in the fund but, to the extent possible, move his money away from the fund during periods when he anticipates a sharp increase in gasoline prices. On January 1, Mr. Haywood deposits $100,000 in the fund. On March 31 his balance is $102,000, and he withdraws $50,000. On May 1 his balance is $52,500 and he deposits $50,000. At the end of the year Mr. Haywood’s fund balance is $111,000. Find the time-weighted yield for the “Tomorrow Fund” for the year, and show that this is lower than Mr. Haywood’s dollar-weighted yield.

\[
\text{Solution. The balance grows as follows:}
\]

\[
\begin{align*}
&\text{Balance} \\
\$100,000 &\rightarrow \$102,000 \rightarrow \$52,500 \rightarrow \$111,000 \\
&\text{Withdrawal} \rightarrow \$50,000 \rightarrow \$102,000
\end{align*}
\]

Therefore \( i_{tw} = \left[ \left( \frac{111,000}{100,000} \right) \left( \frac{52,500}{102,000} \right) \right]^{\frac{1}{3}} - 1 \approx 0.1153053799. \) The dollar-weighted yield \( i \) satisfies

\[
\$111,000 = \$100,000 (1 + i)^{12} + \$50,000 (1 + i)^{11}.
\]

The reason that Mr. Haywood’s dollar-weighted yield exceeds the Tomorrow Fund’s time-weighted yield is that he timed his deposits and withdrawals well. Had they been poorly timed, his dollar-weighted yield would have been worse than the Fund’s time-weighted yield.

2.8 PROBLEMS, CHAPTER 2

(2.0) Chapter 2 writing problems

(1) [Following Section (2.3)] Consider the equation

\[
\sum_{i=1}^{3} \frac{\$2,000 (1 + i)}{(1 + i)^{1}} = \frac{\$2,000 (1 + i)}{1}.
\]

Describe a financial situation for which this is the associated time 4 equation of value. Give details explaining the sources of any deposits and the reasons for any withdrawals.

(2) [Following Section (2.3)] Consider the equation

\[
\sum_{i=1}^{20} \frac{\$10,000 i^{2}}{1 + i^{2}} = \sum_{i=1}^{20} \frac{\$10,000 i^{2}}{1 + i^{2}}.
\]
Describe a financial situation for which this is the associated time 35 equation of value.

(3) Write an advertisement for an investment fund, giving annual yields earned by the fund over each of five years and an annual time-weighted yield for the period.

(4) Learn about these mutual funds and the portfolio focus of each fund. For each fund, note the annual yield rates over various time periods. Include the period from the date of inception to a recent date and a recent five-year period. Comment on similarities and differences in the performance of the funds. Specify on the causes of any performance discrepancies.

(2.2) Equations of value for investments involving a single deposit made under compound interest

(1) Mr. Lopez deposits $K in an account paying 4% annual effective interest. The balance at the end of three years is $982. Find K.

(2) Marianne deposits $20,000 in a five-year certificate of deposit. At maturity the balance is $27,520.64. Find the annual effective rate of interest governing the account.

(3) Suzanne remembers that her only deposit into her savings account was a $1,800 deposit. She knows that the account has a constant nominal interest rate of 2.5% convertible monthly and that the balance is now $1,965.35. How long ago did Suzanne make her deposit?

(4) Use the rule of seventy to approximate the length of time it takes money to double at an annual effective interest rate of 5% and then at an annual effective rate of 10%. Then find the exact time it takes for money to double at each of these interest rates.

(5) Derive a “rule of a” to approximate the length of time it takes for money to triple. As in the derivation of the “rule of seventy-two,” your rule should be derived to give an especially good estimate when the annual effective interest rate is 8%. After you have stated your rule, compare the approximations it gives for annual effective interest rates of 4% and 12% with the true values at these rates.

(2.3) Equations of value for investments with multiple contributions

(1) Sidney borrows $12,000. The loan is governed by compound interest and the annual effective rate of discount is 8%. Sidney repays $4,000 at the end of one year, X at the end of two years, and $3,000 at the end of three years in order to exactly pay off the loan. Find X.

(2) Rafael opens a savings account with a deposit of $1,500. He deposits $100 one year later and $1,000 a year after that. Just after Rafael’s deposit of $1,000, the balance in his account is $3,078. Find the annual effective interest rate governing the account.

(2) Esteban borrows $20,000, and the loan is governed by compound interest at an annual effective interest rate of 6%. Esteban agrees to repay the loan by making a payment of $10,000 at the end of T years and a payment of $12,000 at the end of 2T years. Find T.

(4) Shakari opens a savings account with a deposit of $3,500. She deposits $500 six months later and $800 nine months after opening the account. The balance in Shakari’s account one year after she opened it is $5,012. Assuming that the account grows by compound interest at a constant annual effective interest rate i, find i.

(5) A loan is negotiated with the lender agreeing to accept $8,000 after one year, $9,000 after two years, and $20,000 after four years in full repayment of the loan. The loan is renegotiated so that the borrower makes a single payment of $37,000 at time T and this results in the same total present value of payments when calculated using an annual effective interest rate of 5%. Estimate T using the method of equated time. Also find T exactly.

(6) Anne and Frank Smith each borrow $12,000 from their father. Anne and Mr. Smith have agreed that she will repay her loan in full by paying $5000 in two years and $8000 in four years. Frank prefers to make one lump payment of $15,000 to fully repay his loan. When should he make that payment so that he and his sister each will have the same effective interest rate?

(7) Let \( b_1, b_2, \ldots, b_n \) be positive real numbers. Set \( A = \left( \frac{\sum_{i=1}^{n} b_i}{n} \right) \), the arithmetic mean of the numbers, and \( G = \left( \prod_{i=1}^{n} b_i \right)^{1/n} \), the geometric mean of the numbers. The objective of this problem is to establish that \( A \geq G \) and that \( A > G \) whenever \( b_1, b_2, \ldots, b_n \) are not all equal.

(a) Write the point-slope equation for the tangent line to \( y = \ln x \) at \((A, \ln A)\).

(b) Use (a) and concavity to show that \( \ln x = A^{-1}(x - A) + \ln A \) for all positive \( x \). Moreover, show that this equality holds if and only if \( x = A \).

(c) Use (b) to prove that \( \ln G = \frac{1}{n} \sum_{i=1}^{n} A^{-1}(b_k - A) + \ln A \) and that this is a strict inequality unless all the \( b_k \)'s are equal.

(d) Show that \( \frac{1}{n} \sum_{k=1}^{n} A^{-1}(b_k - A) + \ln A = \ln A \).

(e) Conclude that \( G < A \) and \( G < A \) unless all the \( b_k \)'s are equal.
(8) Let \(C_k\) denote the contribution in cents made at distinct times \(k, k = 1, 2, \ldots, n\). Suppose that these are all positive so that we have deposits, but no withdrawals. Then the \(C_k\)'s are positive integers. As in (2.3.9), let \(T = \ln\left(\sum_{k=1}^{n} C_k \frac{v^k}{C}\right)/\ln v\). As in (2.3.10), let \(T' = \sum_{k=1}^{n} (C_k / v^k)\). The objective of this problem is to use the result of Problem (2.3.7) to establish that \(T' > T\) and that this is a strict inequality unless \(n = 1\).

(a) Consider \(C_k\) quantities each equal to \(a^k, k = 1, 2, \ldots, n\). Then in all we are considering \(C = C_1 + C_2 + \cdots + C_n\) quantities. Use the result of Problem (2.3.7)(c) to show that

\[
C_1 v^1 + C_2 v^2 + \cdots + C_n v^n \geq v^n
\]

and that this is a strict inequality unless \(n = 1\).

(b) Use (a) to show that the present value of the deposits is at least as large as the present value given by the method of equated time and that it is strictly larger unless \(n = 1\).

(c) Show that \(T' > T\) with strict inequality unless \(n = 1\).

(9) Suppose that you pay $1,000 at time 0, get $4,000 at time 1, and pay $2,000 at time 2. Let \(C_0 = 1,000, C_1 = -4,000,\) and \(C_2 = 2,000.\)

(a) Find \(T\) such that getting an inflow of \(-C\) at time \(T\) has the same present value as the above sequence of financial transactions, assuming that the growth of money is governed by compound interest at \(i = 1\%\). Show that \(T\) is greater than the weighted average \(\frac{T_0}{C_0} + \frac{T_1}{C_1} + \frac{T_2}{C_2}\). [This shows that Inequality (2.3.11) need not hold if you have a negative contribution.]

(b) [recommended for those with a BA II Plus calculator] Dax borrows $300,000 and the loan is governed by compound interest at an annual effective interest rate of 4.75%. Dax agrees to repay the loan by ten equally spaced payments, the first four of which are for $25,000 and the next six of which are for $40,000. When should he make the first payment?

(c) Find the amount to be paid at the end of eight years that is equivalent to a payment of $400 now and a payment of $300 at the end of four years:

(a) if 5% simple interest is earned from the date each payment is made and use a comparison date of right now.

(b) if 5% simple interest is earned from the date each payment is made and use a comparison date of eight years from now.

(c) Explain why the fact you get different answers in parts (a) and (b) does not contradict the fact that equations of value at different times are equivalent.

(d) Repeat parts (a) and (b) except replace “simple interest” with “compound interest.”

(12) [calculator needed] Use Newton's method to solve the problem of Example (2.3.7). More specifically, let \(f(t) = 525(1.1)^{-t} + 525(1.1)^{-t} - 1,000\), make an initial guess \(T_1\) for a root \(T\), and find a sequence of approximations \(\{T_k\}\) to \(T\) that allow you to obtain \(T\) to the nearest hundredth of a percent. Why might \(T_1 = .3\) be a reasonable initial guess?

(2.4) Investment return

1. Payments of $3,000 now and $6,000 eight years from now are equivalent to a payment of $4,000 four years from now at either rate \(i\) or rate \(j\). Find \(i - j\). Explain why the yield rate is not unique in this case.

2. Success, Inc. enters into a financial arrangement in which it agrees to pay $100,000 today and $100,000 two years from now in exchange for $200,000 one year from now. Show that there is no yield rate that can be assigned to this two-year transaction.

3. Signum, Inc. agrees to pay $500,000 today and $40,000 four years from today in return for $210,000 two years from today. What is the yield rate for this four-year financial arrangement?

4. Firms A, B, C, and D enter into a financial arrangement. Money flush firm A will pay expanding firms B and C each $1,000,000 today. B will pay D $2,200,000 three years from today. C will pay B $800,000 two years from today and D $550,000 two years from today. Finally, D will pay A $3,200,000 six years from today. Calculate the yield rate or interest rate, to the nearest hundredth of a percent, that each firm experiences over the period of their involvement (6 years for A, 3 years for B, 2 years for C, and 4 years for D).

5. Stella invests $8,572.80 at \(r_1 = 0\) and $28,500 at \(r = 2\). In return, she receives $27,074 at \(t = 1\) and $10,000 at \(t = 3\). Write down a time 0 equation of value and verify that it is satisfied for \(v = 94, v = .95,\) and \(v = .96.\) Find the corresponding three yield rates.

6. Pedro invests $100,000 at \(r_1 = 0\) and $50,000 at \(r = 2\). In return he gets $60,000 at \(t = 1\) and $126,500 at \(t = 3.\) Write down a time 3 equation of value describing Pedro's investment. Explain why there is a unique yield rate and find it.
(7) Show that if $1 < j \leq i$, then $\sum_{t=1}^{j} C_t (1 + i)^{t-i} = \sum_{t=1}^{j} C_t (1 + j)^{t-j}$, and $B_j(i)$, $B_k(i)$, ..., $B_{i-1}(i)$ are all positive (with $B_k(i)$ as given in (2.4.8)), then $j = i$.

(8) [recommended for those with a BA II Plus calculator]
On January 1, Ezekiel opens an account at Friendly Bank. His opening deposit is for $500 and he makes deposits at the end of each quarter for four years, then makes no more deposits. He closes the account exactly seven years after he opens it and receives $3423.28. Find his annual yield rate for this seven-year period if his quarterly deposits were $50 in the first year, $75 in the second year, $50 in the third year, and were successively $300, $450, $600, and $240 in the fourth year.

(9) [recommended for those with a BA II Plus calculator]
A loan of $20,000 is to be repaid by thirty-three end-of-month payments. The first payment is $400 and each subsequent payment is $35 more than the previous payment. Find the annual yield rate correct to the nearest hundredth of a percent. Hint: The Cash Flow worksheet only accepts twenty-four payments, or thirty-two if you have a BA II Plus Professional calculator. If you are working with the BA II Plus calculator, suppose that the payments beyond the twenty-fourth, which you do not have registers to accommodate, are made along with the twenty-fourth. Now use the "guess and check" method, obtaining your first estimate by using IRR or CPT. This is a challenging problem, especially for those of you with only 24 registers. However, when performing the successively calculated rates by the "guess and check" method, you may make judicious use of the NPV worksheet to decrease your work.

(2.5) Reinvestment considerations

(1) Angela loans Kathy $8,000. Kathy repays the loan by paying $5,000 at the end of one year and $4,000 at the end of three years. The money received at $r = 1\frac{1}{2}$ is immediately reinvested at an annual effective interest rate of 6%. Find Kathy's annual effective rate of interest and Angela's annual yield.

(2) Kurt loans Randy $14,000. Randy repays the loan by paying Kurt $4,000 at the end of one year and $4,000 at the end of two years and as well as at the end of three years. The money received at $r = 1$ and at $r = 2$ is immediately reinvested at an annual effective interest rate of 3%. Correct to the nearest tenth of a percent, find Randy's annual effective interest rate and Kurt's annual yield.

(3) On January 15, 2000, Enterprise A loans $6,000 to Enterprise B and $17,000 to Enterprise C. Enterprise B repays Enterprise A $7,000 on January 15, 2002 and this money is reinvested at a 5% annual effective rate. Enterprise C repays Enterprise A $22,500 on January 15, 2004. What is the annual yield received by Enterprise A over the four-year interval. Compare it to the annual effective interest rates paid by Enterprises B and C.

(2.6) Approximate dollar-weighted yield rates

(1) Sandra invests $10,632 in the Wise Investment Fund. Three months later her balance has grown to $11,902 and she deposits $2,500. Two months later her fund holdings are $14,308 and she withdraws $7,000. Two years after her initial investment, she learns that her holdings are worth $12,596.

(a) Write an equation of value involving the exact dollar-weighted annual yield $\tau$ over the two-year period.
(b) Compute the approximate dollar-weighted annual yield for the investment period using (2.6.5) and again using (2.6.8).

(2) On February 1, Arshak’s investment account has a balance of $19,800. He deposited $1,200 on April 1 and $2,600 on May 1. He withdrew $8,400 on July 1. On November 1, Arshak’s balance was $14,820. Find Arshak’s approximate dollar-weighted annual yield for this nine-month period using (2.6.2).

(3) Franklin’s investment fund had a balance of $290,000 on January 1, 1995 and a balance of $446,000 two years later. The amount of interest earned during the two years was $36,000, and the annual yield rate on the fund was 5.4%. Estimate the average dollar-weighted yield of contributions to the account.

(4) The investment balance of a firm is $5,000,000 at the beginning of a two-year period and $7,600,000 at the end. The firm makes a single contribution during the two-year interval of $1,300,000. What is the difference between the approximate annual dollar-weighted yield earned by the firm if the contribution is made after 6 months as opposed to it being made after one year?

(2.7) Fund performance

(1) On January 1, 1999, Antonio invests $9,400 in an investment fund. On January 1, 1999 his balance is $10,600 and he deposits $3,400. On July 1, 1999 his balance is $14,400 and he withdraws $1,000. On January 1, 1992 his balance is $3P. Express his annual time-weighted yield as a function of $P$.

(2) Arthur buys $2,000 worth of stock. Six months later, the value of the stock has risen to $2,300 and Arthur buys another $1,000 worth of stock.
After another eight months, Arthur's holdings are worth $2,700 and he sells off $600 of them. Ten months later, Arthur finds that his stock has a value of $2,100.

(a) Compute the annual time-weighted yield rate of the stock over the two-year period.

(b) Compute the annual dollar-weighted yield for Arthur over the two-year period.

(c) Should the answer in part (a) or part (b) be larger? Why?

(3) Bright Future Investment Fund has a balance of $1,205,000 on January 1. On May 1, the balance is $1,230,000. Immediately after this balance is noted, $800,000 is added to the fund. If there are no further contributions to the fund for the year and the time-weighted annual yield for the fund is 16%, what is the fund balance at the end of the year?

Chapter 2 review problems

(1) Sahil makes an initial investment of $20,000. In return, he receives $4,000 at the end of one year and another $10,000 at the end of three years.

(a) Assuming that the investment is made at simple interest at rate r, write down an equation for the investment and find r.

(b) Assuming that the investment is made at compound interest at effective interest rate i, write down an equation for the investment and justify the statement that there is a unique yield rate. Use the “guess and check method” to estimate i to the nearest hundredth of a percent.

(c) Starting with the same initial guess for i that you used in (b), check your answer using Newton's method.

(d) (Recommended for those with a BA II Plus calculator) Use the Cash Flow worksheet to find I for the nearest millionth.

(2) Suppose now that investments are governed by compound interest at an effective interest rate i > 0. By how much does the sum of the time n value of $K$ paid at time 0 and the time n value of $K$ paid at time 2n exceed $2K$? Express your answer as a function $g(n)$ and show that $g(n) > 0$ if $n > 0$.

(b) Suppose that investments are governed by the simple interest accumulation function $a(r) = 1 + r$, $r > 0$. Does the sum of the time n value of $K$ paid at time 0 and the time n value of $K$ paid at time 2n exceed $2K$ for all r and n? Justify your answer.

(3) Elise invests $16,312 at r = 0. In return, she gets $50,000 at t = 1 and $10,000 at t = 2. Half of the time 1 payment, she reinvests at an annual effective interest rate of 5%. What is her annual yield rate for the two-year period?

(4) Sports Manufacturing needs capital for expansion. It borrows $1,000,000 from Venture Bank for three years at 5% nominal interest convertible quarterly, and $500,000 for five years from a private investor at a 5% effective discount rate. At the end of two years, Sports Manufacturing makes a $200,000 three-year loan to its supplier of titanium (for baseball bats) at 7% annual effective interest. What annual internal rate of return should Sports Manufacturing report for the combined cashflows over the five-year period?

(5) Abiyote invested $24,500 on January 1, 1994 in the Utopia Fund. On May 1, 1995, his balance was $38,212 and he withdrew $10,000. On December 1, 1995, his balance was $15,892, and he deposited $8,000. On January 1, 1997 his balance was $30,308.

(a) Find the annual time-weighted yield for the Utopia Fund for the three-year period from January 1, 1994 until January 1, 1997.

(b) Find an approximate annual dollar-weighted yield received by Abiyote for the three-year period from January 1, 1994 until January 1, 1997 using (2.6.5).

(c) (Recommended for those with a BA II Plus calculator) Find the dollar-weighted yield received by Abiyote for the three-year period from January 1, 1994 until January 1, 1997, correct to the nearest millionth of a percent.

(d) Compare the time-weighted yield experienced by the Utopia Fund and the dollar-weighted yield received by Abiyote from his investment in the Utopia Fund. Discuss why the inequality between them is in the direction it is.

(6) Xiang and Dmitry are friends. They agree that Xiang will pay Dmitry $500 immediately and another $500 at the end of three years. In return, Dmitry will pay Xiang $K$ in exactly one year and again at the end of exactly two years.

(a) Find K if the transaction is based on compound interest at a nominal discount rate of 6% convertible monthly.

(b) If K = 600, is there a unique positive yield rate for the transaction? Justify your answer.