(1) Write out the addition and multiplication tables for \( \mathbb{Z}_4 \) (that is, addition and multiplication mod 4). Find an field axiom that fails.

- In \( \mathbb{Z}_4 \), the element 2 does not have a multiplicative inverse:
  
  \[
  0 \cdot 2 = 0, \quad 1 \cdot 2 = 2, \quad 2 \cdot 2 = 0, \quad 3 \cdot 2 = 2.
  \]

(2) What would happen if we relax the restriction that the identity elements in a field be distinct?

- Under the new axioms there is exactly one more field, it has one element, say \( 'e' \) and

  \[
  e + e = e \quad \text{and} \quad e \cdot e = e.
  \]

  The proof is as follows: Suppose \( F \) contains an element \( a \neq 0 \). Then

  \[
  a \cdot 0 + 0 = a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0
  \]

  by subtracting \( a \cdot 0 \) from both sides we obtain \( a \cdot 0 = 0 \). But \( a \cdot 1 = a \neq 0 \) hence \( 1 \neq 0 \). This shows that if \( F \) is a field under the new axioms, then either (a) \( F \) contains only one element, or (b) \( 0 \neq 1 \) and so \( F \) is a field under the old axioms.

(3) Let \( V \) be a vector space over a field \( F \) show that for all \( a, b \in F \) and all \( x, y \in V \),

  \[
  (a + b)(x + y) = ax + ay + bx + by
  \]

  quoting the appropriate axiom for each step.

- Easy.

(4) Let us write \( 0 \) for the zero vector. If \( a \) is an element of the field, show that \( a0 = 0 \).

- Well,

  \[
  a0 + a0 = a(0 + 0) = a0
  \]

  and so subtracting \( a0 \) from both sides gives \( a0 = 0 \).

(5) Find a subset of \( \mathbb{R}^3 \) which is closed under scalar multiplication, but not vector addition.

- It is not difficult to check that the following works:

  \[
  \left\{ \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} : x \in \mathbb{R} \right\} \cup \left\{ \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix} : x \in \mathbb{R} \right\}
  \]
(6) Exercise 23 from §1.3.
• (a) As $W_1$ and $W_2$ are subspaces, each contains 0. Hence any element $w_1 \in W_1$ belongs to $W_1 + W_2$ because it can be written as $w_1 + 0$. This shows $W_1 \subseteq W_1 + W_2$.

That $W_2 \subseteq W_1 + W_2$ follows by the same argument with roles reversed.

To see that $W_1 + W_2$ is a subspace, we notice that for all $w_1, v_1 \in W_1$, all $w_2, v_2 \in W_2$, and all $c \in F$ we have

$$(w_1 + w_2) + (v_1 + v_2) = (w_1 + v_1) + (w_2 + v_2) \in W_1 + W_2$$

and

$$c(w_1 + w_2) = (cw_1) + (cw_2) \in W_1 + W_2.$$  

• (b) Let $W$ be a subspace of $V$ that contains both $W_1$ and $W_2$. We need to show that $W_1 + W_2$ is a subset of $W$. Well given $w_1 \in W_1$ and $w_2 \in W_2$ we know that both belong to $W$; moreover since $W$ is a subspace, $w_1 + w_2 \in W$. In this way we have shown that each element of $W_1 + W_2$ belongs to $W$, that is, $W_1 + W_2 \subseteq W$.

(7) Exercises 4(a) and 5(g) from §1.4.
• 4(a) Yes:

$$x^3 - 3x + 5 = 3(x^3 + 2x^2 - x + 1) - 2(x^3 + 3x^2 - 1).$$

• 5(g) Yes:

$$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} = 3 \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$