(1) Let $W$ and $V$ be finite dimensional vector spaces and let $T : V \rightarrow W$ be linear. Show that

(a) If $\dim(V) < \dim(W)$ then $T$ is not onto.
(b) If $\dim(V) > \dim(W)$ then $T$ is not one-to-one.

(2) Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}$ is linear. Show that there is a vector $a \in \mathbb{R}^3$ so that

$$T(v) = a \cdot v$$

with the usual dot product from calculus.

(3) Suppose $T : V \rightarrow V$ is linear and one-to-one. Show that for any ordered basis $\beta$ of $V$, there is a basis $\gamma$ of $V$ so that $[T]_\beta^\gamma$ is diagonal.

(4) Suppose $T : V \rightarrow V$ is linear. Show that $T^2$ is the zero transformation if and only if $R(T) \subseteq N(T)$.

(5) Find $2 \times 2$ matrices $A$ and $B$ so that $AB$ is the zero matrix, but $BA$ is not.

(6) Let $P_n = \{a_0 + a_1 x + \cdots + a_n x^n : a_0, \ldots, a_n \in \mathbb{R}\}$.

(a) Find the matrices of

$$T : P_3 \rightarrow P_4 \quad \text{by} \quad P(x) \mapsto \int_0^x P(t) \, dt$$

and

$$D : P_4 \rightarrow P_3 \quad \text{by} \quad P(x) \mapsto P'(x) \quad \text{(differentiation)}$$

with respect to the natural bases.

(b) Determine the matrices for $TD$ and $DT$.

(7) Let $W$ and $V$ be vector spaces and let $T : V \rightarrow W$ and $S : W \rightarrow V$ be linear transformations. Suppose further that $ST$ is the identity transformation:

$$ST(v) = v$$

(a) Show that $\dim(W) \geq \dim(V)$.
(b) Show that if $\dim(W) = \dim(V)$, then $T$ is invertible.