Homework 2
Math 270C: Applied Numerical Linear Algebra
Due: Friday, May 27.

1 Pen and Paper Problems

1. Prove that \( \text{span}\{p_1, p_2, \ldots, p_k\} = \text{span}\{p_1, A^T p_1, (A^T)^2 p_1, \ldots, (A^T)^{k-1} p_1\} \) for the unsymmetric Lanczos iteration (you may assume there is no breakdown, catastrophic or otherwise).

2. Show that unsymmetric Lanczos is equivalent to the two sided Graham-Schmidt process applied to the columns of the matrices

\[
F = [q_1, Aq_1, A^2 q_1, \ldots, A^{n-1} q_1], \quad G = F^T
\]

3. Consider the matrix

\[
\frac{1}{\Delta x^2} \begin{pmatrix}
2 & -1 & 0 & 0 & \ldots & 0 \\
-1 & 2 & -1 & 0 & \ldots & 0 \\
0 & -1 & 2 & -1 & \ldots & 0 \\
\vphantom{0} & \vphantom{0} & \vphantom{0} & \vphantom{0} & \ddots & \ddots \\
0 & \vphantom{0} & \vphantom{0} & \vphantom{0} & \vphantom{0} & -1 & 2
\end{pmatrix} \in \mathbb{R}^{(n-2) \times (n-2)}, \quad \Delta x = \frac{1}{N-1}.
\]

Show that its eigenvectors \((x^k)\) are given by

\[
x_j^k = \sin(k \pi j \Delta x)
\]

and derive a formula for its eigenvalues (use the identity \(\sin(k \pi j \Delta x) = \frac{e^{ik \pi j \Delta x} - e^{-ik \pi j \Delta x}}{2i}\)).

2 Programming problems

In the problems that follow, you will need to implement and test MINRES on the appropriate matrix from the previous assignment. You should solve the discrete PDE with \(N = 65,129,257,513\) (and for all cases, use 64 iterations of the given method). Submit a plot that will show me that you have implemented the discrete elliptic PDE correctly. I.e. plot \(\log(||e||_\infty)\) vs. \(\log(N)\) (where \(e^N \in \mathbb{R}^N\) is the error of the discretization of the PDE: \(e^N_i = \sin(2 \pi x_i) - u^N_i\) and \(u^N_i\) is from the discrete problem with resolution \(N\)). Also include a plot of the best-fit line to this data and report the slope. The second plot should be \(\log(||r||_2)\) vs. \(k\) and should contain this data for each of the choices of grid resolution \(N = 65,129,257,513\) (use color coding and a legend to distinguish which is which). In general, the condition number of the discrete PDE will be expected to get worse with \(N\) so you should expect to see slower convergence (in terms of number of iterations required) for the higher resolution problems. In all cases, use an initial guess of \(u_0 = 0\).

1. Implement the MINRES (version 1) algorithm and test it on the symmetric indefinite matrices for the discrete problem described in hw1.

2. Implement the MINRES (version 2) algorithm and test it on the symmetric indefinite matrices for the discrete problem described in hw1.
3. Implement GMRES(m). This is the version of GMRES that restarts with a new initial guess after \( m \) iterations. Run GMRES(1) and GMRES(2) on the following problem and comment on your results:

\[
A = \begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 3 \\
0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
2 \\
-4 \\
1
\end{pmatrix}
\]

4. Implement the unsymmetric Lanczos iteration to factorize (or attempt to in the case of catastrophic breakdown) the non-symmetric matrix in the last assignment. Use the right hand side of that problem to generate the first two basis vectors. Also, run with \( N = 33 \).