1 Theory

1. (Strikwerda 5.1.2.) Show that the modified leapfrog scheme (5.1.6) is stable for $\epsilon$ satisfying

$$0 < \epsilon \leq 1 \quad \text{if} \quad 0 < a^2 \lambda^2 \leq \frac{1}{2}$$

and

$$0 < \epsilon \leq 4a^2 \lambda^2 (1 - a^2 \lambda^2) \quad \text{if} \quad \frac{1}{2} \leq a^2 \lambda^2 < 1.$$ 

Note that these limits are not sharp. It is possible to choose $\epsilon$ larger than these limits and still have the scheme be stable.

2. Derive the stability condition for the backward-time forward-space scheme

$$\frac{1}{k} (v_{m+1} - v_m^n) + \frac{a}{h} (v_{m+1}^{n+1} - v_{m}^{n+1}) = 0$$

used to approximate solutions to $u_t + au_x = 0$ with, say, $x \in [0, 1]$ and periodic boundary conditions. Give an example of an initial condition $v_0^m$ and an explicit expression for $v_n^m$ that demonstrate unstable behavior for a particular $\lambda$ (your choice) which fails to satisfy the stability condition. Does the growth in your example agree with your theoretical amplification factor?

3. Prove that numerical solutions to the Lax-Friedrichs scheme

$$\frac{1}{k} \left( v_{m+1}^{n+1} - \frac{1}{2} (v_{m+1}^n + v_{m+1}^{n+1}) \right) + \frac{a}{2h} (v_{m+1}^n - v_{m-1}^n) = 0$$

converge to solutions to the corresponding modified equation

$$u_t + au_x = \frac{h^2}{2k} \left( 1 - \left( \frac{ak}{h} \right)^2 \right) u_{xx}$$

to second order accuracy in $\ell^\infty$. I.e., show that $|v_n^m - u_{k,h} (n, x_m)| \to 0$ as $h, k \to 0$ (according to the stability criterion), where the subscripts on $u_{k,h}$ only indicate that the solution to the modified equation is parameterized by $k, h$.

4. (Strikwerda 4.1.2.) Show that the $(2,2)$ leapfrog scheme for $u_t + au_{xxx} = f$ (see (2.2.15)) given by

$$\frac{v_{m+1}^n - v_m^{n-1}}{2k} + a\delta^2 \delta_0 v_m^n = f_m^n,$$

with $\nu = k/h^3$ constant, is stable if and only if

$$|\nu| < \frac{2}{3^{3/2}}.$$
5. (Strikwerda 3.2.1.) Show that the (forward-backward) MacCormack scheme

\[
\tilde{v}_{m}^{n+1} = v_{m}^{n} - a\lambda (v_{m+1}^{n} - v_{m}^{n}) + kf_{m}^{n}, \\
v_{m}^{n+1} = \frac{1}{2} (v_{m}^{n} + \tilde{v}_{m}^{n+1} - a\lambda (\tilde{v}_{m+1}^{n+1} - \tilde{v}_{m-1}^{n+1}) + kf_{m}^{n+1})
\]

is a second-order accurate scheme for the one-way wave equation (1.1.1). Show that for \( f = 0 \) it is identical to the Lax-Wendroff scheme (3.1.1).

2 Programming

1. For the one-way wave equation \( u_t + au_x = 0 \), investigate how close the numerical solution to a finite difference scheme is to the solution to the corresponding modified equation. To be concrete, suppose a pulse initial condition \( u_0(x) = \frac{1}{2} (1 + |x|/x) \), \( x \in [-1, 1] \), and periodic boundary conditions. Take \( a = 1 \), \( k/h = 0.5 \), and final time \( T = 0.5 \). Compare the following finite difference schemes: upwinding, Lax-Friedrichs, and Lax-Wendroff. Also, include a derivation of the respective corresponding modified equations. You may find solutions to the modified equations using any appropriate method (i.e., analytically or to a sufficiently high accuracy numerically).