1 Theory

1. (Strikwerda 2.1.9.) Finite Fourier Transforms. For a function \( v_m \) defined on the integers, \( m = 0, 1, \ldots, M - 1 \), we can define the Fourier transform as

\[
\hat{v}_\ell = \sum_{m=0}^{M-1} e^{-2i\pi \ell m/M} v_m \quad \text{for } \ell = 0, \ldots, M - 1.
\]

For this transform prove the Fourier inversion formula

\[
v_m = \frac{1}{M} \sum_{\ell=0}^{M-1} e^{2i\pi \ell m/M} \hat{v}_\ell,
\]

and the Parseval’s relation

\[
\sum_{m=0}^{M-1} |v_m|^2 = \frac{1}{M} \sum_{\ell=0}^{M-1} |\hat{v}_\ell|^2.
\]

Note that \( v_m \) and \( \hat{v}_\ell \) can be defined for all integers by making them periodic with period \( M \).

2. Prove convergence for the Beam-Warming scheme

\[
u_{m}^{n+1} = u_m^n - \frac{ak^2}{2h} \left( 3u_m^n - 4u_{m-1}^n + u_{m-2}^n \right) + \frac{a^2k^2}{2h^2} \left( u_m^n - 2u_{m-1}^n + u_{m-2}^n \right)
\]

used to approximate solutions to \( u_t + au_x = 0 \) for \( a > 0 \).

3. (Strikwerda 2.2.4.) Show that the box scheme

\[
\frac{1}{2k} \left( \left( v_m^{n+1} + v_{m+1}^{n+1} \right) - \left( v_m^n + v_{m+1}^n \right) \right) + \frac{a}{2h} \left( \left( v_{m+1}^{n+1} - v_{m}^{n+1} \right) + \left( v_{m+1}^n - v_{m}^n \right) \right) = f_m^n
\]

is consistent with the one-way wave equation \( u_t + au_x = f \) and is stable for all values of \( \lambda \).

4. (Strikwerda 2.2.6.) Determine the stability of the following scheme, sometimes called the Euler backward scheme, for \( u_t + au_x = f \):

\[
v_m^{n+1/2} = v_m^n - \frac{a\lambda}{2} \left( v_{m+1}^n - v_{m-1}^n \right) + kf_m^n,
\]

\[
v_m^{n+1} = v_m^n - \frac{a\lambda}{2} \left( v_{m+1}^{n+1/2} - v_{m-1}^{n+1/2} \right) k f_{m+1}^{n+1}.
\]

The variable \( v^{n+1/2} \) is a temporary variable, as is \( \tilde{v} \) in Example 2.2.5.
2 Programming

1. (Strikwerda 2.3.3.) Solve the initial value problem for equation

\[ u_t + \left( 1 + \frac{1}{4} (3 - x) (1 + x) \right) u_x = 0 \]

on the interval \([-1, 3]\) with the Lax-Friedrichs scheme (2.3.1) with \(\lambda\) equal to 0.8. Demonstrate that the instability phenomena occur where \(|a(t,x)\lambda|\) is greater than 1 and where there are discontinuities in the solution. Use the same initial data as in Exercise 2.3.1. Specify the solution to be 0 at both boundaries. Compute up to the time of 0.2 and use successively smaller values of \(h\) to show the location of the instability.

2. Investigate (via numerical evidence) the convergence (or lack thereof) of the forward-time central-space scheme

\[ \frac{1}{k} \left( u_{m+1}^{n+1} - u_m^n \right) + \frac{a}{2h} \left( u_{m+1}^n - u_{m-1}^n \right) = 0 \]

in the \(L^\infty\)-norm. Use the same scenarios from Homework 1, e.g., compare your results using smooth, continuous-but-non-smooth, and discontinuous initial conditions. Be sure to restrict the relation between \(k\) and \(h\) appropriately. Compare convergence in the \(L^\infty\)-norm with convergence in the \(L^2\)-norm.